Math
QA
485
J13

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY

Cornell University Library

THE EVAN WILHELM EVANS
MATHEMATICAL SEMINARY LIBRARY

THE GIFT OF
LUCIEN AUGUSTUS WAIT

M.106
MATHEMATICS
31|V|10
7348-1

Cornell University Library
QA 485.J13

Geometrical conic sections; an elementary

3 1924 001 105 497 math

Cornell University Library

The original of this book is in
the Cornell University Library.

There are no known copyright restrictions in
the United States on the use of the text.

http://www.archive.org/details/cu31924001105497

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY

GEOMETRICAL CONIC SECTIONS.

AN ELEMENTARY TREATISE

IN WHICH THE CONIC SECTIONS ARE DEFINED AS THE PLANE SECTIONS OF A CONE, AND TREATED BY THE METHOD OF PROJECTIONS.

BY

J. STUART JACKSON, M.A.,
LATE FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

London and New York:
MACMILLAN AND CO.
1872.

[*All Rights reserved.*]

P

M.106

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

THE following pages have been written with a view to give the student the benefit of the Method of Projections as applied to the Ellipse and Hyperbola. This Method is calculated to produce a material simplification in the treatment of those curves and to make the proof of their properties more easily understood in the first instance and more easily remembered. It is also a powerful instrument in the solution of a large class of problems relating to these curves.

When the Method of Projections is admitted into the treatment of the Conic Sections there are many reasons why they should be defined, not as has been the case of late years with reference to the focus and directrix, but according to the original definition from which they have their name, as plane Sections of a Cone. First and principally, because this definition gives an immediate proof of the relation by projection of the Ellipse and Circle and of the General and Rectangular Hyperbolas: and in the second place, it naturally divides the properties that may be proved by projection from those connected with the focus and directrix, and thus introduces a valuable simplification into the treatment of the subject. It is also a consideration of some importance that we can see at once from the form of the cone, the general form of the curves that may be cut from it by a plane in different positions; and, by turning the plane about a certain line, we see how the curves pass from one form into another.

It is hoped these may be thought sufficient reasons for departing from the definition which has been in use of late years.

I have retained the proof of the constancy of the ratio of the rectangles under the segments of intersecting chords in constant directions (Chap. V. § 15), given by Dean Hamilton in his treatise published in 1773, as an improving and interesting study and one that is in harmony with the scheme of this work.

I have to acknowledge the great kindness of Rev. Chas. Taylor, Fellow of St John's College, Cambridge, in permitting me the use of his excellent proof of the intersection of tangents at the extremities of a chord of any Conic Section on the diameter of the chord; with the proofs of $CV.CT = CP^2$ (Chap. V. § 8), and $PV=PT$ (Chap. VII. § 20), depending on it: and also of his proof of $SQP=HQP'$ (Chap. IV. § 13).

Professor Adams' property of the tangent is indispensable in any geometrical treatment of the Conic Sections, and I have his kind permission to make use of it.

I shall be greatly obliged by any corrections and hints towards improvement in case this work should be so fortunate as to reach a second edition.

9, BROOKSIDE, CAMBRIDGE,
Christmas, 1871.

TABLE OF CONTENTS.

CHAPTER I.

ON THE METHOD OF PROJECTIONS.

ART. PAGE

1. Definitions 1
2. Construction 1
3—5. Relation between lines and their projection 2
6. Effect of projection on angles 4
7. ” ” ” areas 5
8. Tangents project into tangents 5
9. Object of the Method 6

CHAPTER II.

ON THE CONE AND SPHERE.

1, 2. Definitions, &c. 7
3. Every plane section of a sphere is a circle 8
4, 5. Tangents to a sphere from an external point are equal . 8

CHAPTER III.

ON THE ELLIPSE.

1. Definition 10
2. The ellipse is the projection of a circle 11
3. Minor axis 12
4. Conjugate diameters 13
5. Supplemental chords 14
6. $QV^2 : CD^2 :: PV . Vp : CP^2$ 14

ART. PAGE
7—9. $CV \cdot CT = CP^2$; $CN \cdot CT = CA^2$, &c. 14
10. $PF \cdot PG = BC^2$; $PF \cdot Pg = AC^2$ 15
11. $NG : NC :: BC^2 : AC^2$ 16
12. $CP^2 + CD^2 = AC^2 + BC^2$ 16
13. $QO \cdot Oq : RO \cdot Or$ a constant ratio 17
14. Circumscribing parallelogram 18

CHAPTER IV.

FOCAL PROPERTIES OF THE ELLIPSE.

1, 2. Preliminary 24
3—5. Foci and directrices 25
6. Secant and tangent 29
7. Tangent equally inclined to the focal distances 30
8. Auxiliary circle 31
9. $SY \cdot HZ = BC^2$ 32
10. $SP \cdot HP = CD^2$ 33
11. $ST : QN :: AS : AX$ 33
12—14. Intersecting tangents 34
15. $SG : SP :: SA : AX$ 36

CHAPTER V.

ON THE HYPERBOLA.

1. Definition 42
2. Projection of the rectangular hyperbola 43
3. Magnitude of the axes 46
Properties of the rectangular hyperbola:—
4. Asymptotes 47
5. Intercepts between asymptotes and curve equal. Locus of the middle points of parallel chords 48
6. $QV^2 + CP^2 = VU^2$ 51
7, 8. $CV \cdot CT = CP^2$; $CN \cdot CT = CA^2$. 52
9. Conjugate hyperbola, conjugate diameters 54
10. Inscribed parallelogram 56
11. Intercepts on a line parallel to an asymptote . . . 57

ART. PAGE
12, 13. These properties transferred to the general hyperbola . 58
14. Supplemental Chords 61
15, 16. Ratio of rectangles on the segments of intersecting chord 61
17. $NG : NC :: BC^2 : AC^2$ 65
18. $PF . PG = BC^2$; $PF . Pg = AC^2$ 65
19. Asymptotes parallel to two generating lines of the cone . 66

CHAPTER VI.

FOCAL PROPERTIES OF THE HYPERBOLA.

(See contents of Chapter IV.)

CHAPTER VII.

ON THE PARABOLA.

1. Definition 87
2. Limiting form of semi-ellipse or semi-hyperbola . . . 88
3. $NP^2 = 4AS . AN$ 89
4. $QR . Q'R = 4AS . PR$ 90
5. Locus of middle points of parallel chords 91
6. $QR^2 = 4AS . PV$ 92
7—9. Focus and directrix 93
10. Secant and tangent 96
11. Tangent equally inclined to the focal distance and the axis of the curve 96
12. Tangents at the extremities of a focal chord . . . 97
13. Perpendicular from the focus on the tangent 97
14—18. Intersecting tangents 98
19. Tangents at the extremities of a chord intersect on its diameter 99
20. $PT = PV$ 99
21. $QV^2 = 4SP . PV$ 100
22. Length of focal chord 101
23. Ratio of the rectangles under segments of intersecting chords 101
24, 25. The normal 102

ERRATA.

Two lines from the bottom of p. 61, after *rectangles* insert *under the segments*.

p. 72, *for* Chap. VII. *read* Chap. VI.

p. 87, *for* Chap. VIII. *read* Chap. VII.

ERRATA.

p. 4, § 6, line 2, *for* except those contained, &c. *read* except those right angles contained by lines parallel and perpendicular to the line of intersection.

p. 14, § 7, line 3, *for* V and T *read* CV and CT.

p. 46, line 12 from the bottom, *for* diminishes the more *read* increases the more.

p. 55, line 4, *for* asymptotes *read* asymptote.

References in Chap. VI. to Chap. V. should be to Chap. IV.

p. 61, line 3 from the bottom, *after* rectangles *insert* under the segments.

p. 72, *for* Chap. VII. *read* Chap. VI.

p. 81, line 10 from the bottom, *for* SQP *read* SQP′.

p. 82, line 8 from the bottom, *for* CR′ *read* CR^2.

p. 87, *for* Chap. VIII. *read* Chap. VII.

p. 91, line 4, *for* Chap. VI. *read* Chap. V.

CONIC SECTIONS.

CHAPTER I.

On the Method of Projections.

1. Definitions. The *Orthogonal projection of a point* on a plane is the foot of the perpendicular drawn from the point to the plane.

The plane on which the projection is made is called the *plane of projection.*

The *orthogonal projection of a line* on a plane is the line traced on the plane by a straight line which passes through all successive points of the given line and is always perpendicular to the plane of projection.

In the present treatise when projections are spoken of it may be understood that orthogonal projections are intended.

If a straight line be drawn through two adjacent points of a curve, and the two points move towards one another on the curve until they coincide, the straight line in its final position is said to be a *tangent* to the curve at the point which it has finally in common with it.

2. In order to project a straight line AB (fig. § 5) on a plane abc, we must draw the perpendicular Aa from some

point A of AB to the plane of projection (Euclid XI. 11), then make a plane pass through AB, Aa; it will be perpendicular to the plane of projection (XI. 18). Hence the perpendicular drawn from any point P of AB to ab the intersection of the planes will be perpendicular to the plane of projection; and the straight line ab will be the projection of AB. Hence the projection of a straight line is also a straight line.

3. The projections of parallel straight lines are also parallel.

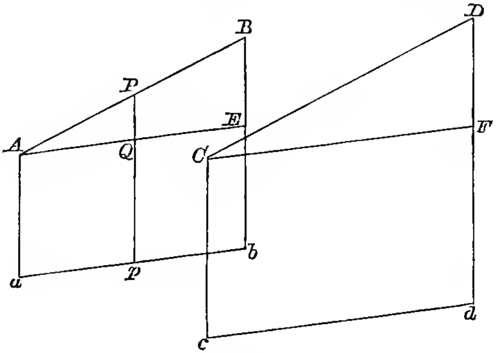
From points A and C (fig. § 5) in the parallel straight lines AB, CD draw Aa, Cc perpendicular to the plane of projection, these lines will be parallel (Euclid XI. 6), and the planes BAa, DCc will be parallel (XI. 15): also the plane of projection will cut these planes in parallel lines ab, cd (XI. 16), and these are the projections of AB, CD. Hence the projections of parallel straight lines are parallel.

4. A line of finite length and its projection are cut in the same ratio by any point and its projection.

Let a, p, b be the projections of the points A, P, B in the straight line AB (fig. § 5), draw AE perpendicular to Bb cutting Pp in Q, then Ap, Qb are parallelograms, and AQ, QE equal to ap, pb. Also PQ is parallel to BE the side of the triangle BAE, and therefore $AQ : QE :: AP : PB$, or $ap : pb :: AP : PB$. The line AB and its projection ab are cut in the same ratio by the point P and its projection p.

5. Parallel straight lines of finite lengths are diminished by projection in the same ratio.

Let the parallel straight lines AB, CD of finite length have projections ab, cd. From A and C draw AE, CF perpendicular to Bb, Dd: these lines are equal to ab, cd: they are also parallel to these parallel straight lines and are therefore parallel to one another (Euclid XI. 9). The lines AB, CD are also parallel, hence the angles BAE, DCF are equal



(XI. 10) : the angles at E and F are right angles, and the triangles BAE, DCF are similar. Hence

$$BA : AE :: DC : CF \text{ or } BA : ab :: DC : cd;$$

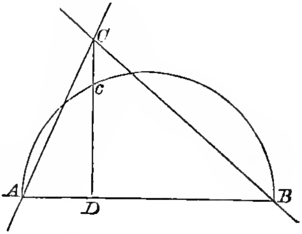
AB, CD are diminished by projection in the same ratio.

Hence it appears that the ratio in which any line is altered by projection from one plane to another depends only upon its inclination to the line of intersection of the planes. It may be seen that lines parallel to the line of intersection are unaltered by projection, and that lines more inclined to it are diminished in a greater ratio than those less so: while lines at right angles to the line of intersection are diminished in the greatest ratio. The ratio which a line at right angles to the line of intersection of its plane with the plane of projection bears to its projection may conveniently be called the *maximum ratio* for the two planes, and it will increase from unity to a ratio as great as we please when we increase the inclination of the planes from zero to a right angle.

In other words, a line at right angles to the line of intersection may have its projection as nearly equal to itself or as small as we please by taking the plane of projection at first very slightly inclined to the plane of the line, and then increasing the inclination till it becomes a right angle. The projection is the base of a right-angled triangle of which the line is the hypotenuse, and the inclination of the planes the angle at the base. By increasing the angle at the base from zero to a right angle we diminish the base from the length of the hypotenuse to zero.

6. Angles will be in general increased or diminished in magnitude by projection, except those contained by lines parallel and right angles perpendicular to the line of intersection.

Of the four angles contained by two intersecting lines, that which is subtended by the line of intersection and the vertical opposite angle will be increased, the other two diminished by projection; and these angles may be increased and diminished to any extent by making the inclination of the plane of projection sufficiently great. If, for example, it be required to increase the acute angle ACB subtended by AB the line of intersection to a right angle; on AB describe the semicircle AcB, and draw CD perpendicular to AB cutting the semicircle in c. Then let the inclination be so chosen that the maximum ratio shall be $CD : cD$: then CD will have its projection equal to cD, and the angle ACB will project into an angle equal to AcB, i.e. to a right angle.



This only holds, however, when D lies between A and B. If D fell outside AB, the angle ACB might be increased or diminished by projection, and would be greatest when the circle circumscribing ABC touched CD, and in that case DC would be a mean proportional between DA and DB.

7. The area of any figure will be reduced by projection in the maximum ratio of the planes.

For every triangle may be divided by a line through one of the angular points parallel to the line of intersection into two, each having its base parallel and its altitude at right angles to the line of intersection: the base of each will be unaltered by projection, the altitude diminished in the maximum ratio. Hence the area of each and therefore of the two together will be diminished in the maximum ratio. Hence, whatever its magnitude or position, every triangle is diminished by projection in the maximum ratio of the planes.

But every figure, whether rectilinear or curvilinear, may be as nearly occupied by triangles as we please, by making them sufficiently numerous. Each of these will be diminished by projection in the maximum ratio of the planes; and therefore the figure composed of them will be diminished in the same ratio.

8. The projection of the tangent to a curve at any point is the tangent to the projection of the curve at the projection of the point.

If p, q be the projections of two points P and Q of a curve, the straight line pq is the projection of the straight line PQ; and if p, q move so as to be always the projections of P and Q as they approach one another on the curve until they coincide and Q is merged in P, p and q will also approach one another on the projected curve, and will finally coincide in the projection of *P*, *PQ* then becomes the tangent to the curve at *P*, and its projection *pq* becomes the tangent to the projection of the curve at the projection of *P*.

9. The object of the method of projections is to extend our knowledge of curves from those which we already know, or can easily investigate, to others into which they project. Thus the properties of the ellipse can be deduced from those of the circle, and the rectangular hyperbola helps us to ascertain the properties of the hyperbola of unequal axes. The preceding propositions serve to connect the diameters and other lines, and the areas, of the known curve with those of its projection.

EXAMPLES.

1. If the inclination of two planes be the third of two right angles, and a line of 4 yards in one of the planes be inclined to the line of intersection at half the above angle, find the area of the square on the projected line.

2. In Art. 6 when *D* falls without *AB*, find the inclination of the planes in order that, in any given case, the angle *ACB* may not be altered by projection.

3. A triangle is projected on to a plane whose line of intersection with its own plane coincides with the base; it is projected back again on to its original plane, and the inclination of the plane is such that after these two projections its vertical angle is unaltered. Shew that this inclination gives the greatest possible increase to the vertical angle at the first projection.

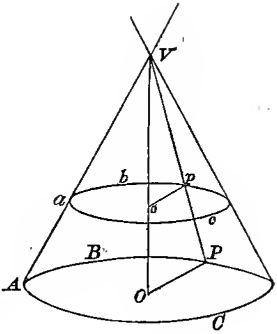
4. Find the altitude of the sun when the greatest shadow of the side of a square on a horizontal plane equals its diagonal.

5. In what position must a cube be placed so that its shadow on a plane that receives the sun's rays directly may be the greatest possible?

CHAPTER II.

On the Cone and Sphere.

1. Definitions. If through O the centre of a circle ABC we draw a straight line OV at right angles to its plane, then the surface traced out by a straight line which passes through any point V of this line and the successive points of the circle ABC is called a *Cone;* or, more distinctively, a *Right Circular Cone*, OV is called the *Axis* of the cone, and V its *vertex.*



We repeat here Euclid's definition of a sphere:

A *sphere* is a solid figure described by the revolution of a semicircle about its diameter which remains fixed.

2. If VP be any position of the generating line of a cone, the angle OVP is the vertical angle of a right-angled triangle of which the height OV and the base OP are constant; therefore OVP is constant, and the angle between the axis and the generating line in all its positions is invariable. Also if any plane at right angles to the axis cuts the cone in the curve abc, abc will be a circle. Let o and p be the points

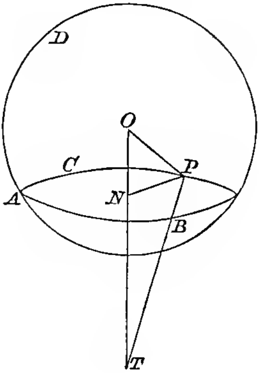
in which the plane cuts the axis and the generating line VP; join op.

Then op is the base of a right-angled triangle of which the height oV and the vertical angle oVp are invariable for all positions of VP; hence op is constant, and the section abc is a circle, with centre o.

Every plane containing the axis cuts the cone in two right lines inclined to each other at a constant angle.

3. From the definition of a sphere it is evident that every point in the sphere is equidistant from the centre of the generating circle.

Every section of a sphere by a plane is a circle.



Let P be any point of the section ABC of a sphere whose centre is O made by a plane. Draw ON at right angles to the cutting plane, and join OP, NP. Then for every point in the section the height ON and the hypotenuse of the right-angled triangle ONP are constant, and therefore the base NP is constant, and ABC is a circle of which N is the centre.

4. Definition. A straight line is said to *touch* a *sphere* when it touches the circle in which the plane through the line and the centre of the sphere cuts the sphere.

It is evident that if a plane be drawn at right angles to the diameter of a sphere through its extremity, every line drawn in this plane through the extremity of the diameter will touch the sphere, and therefore the plane is said to touch the sphere.

5. All the lines drawn from one point to touch a sphere are equal.

Let TP (fig. § 3) be any line drawn from the point T to touch the sphere $ABCD$ whose centre is O in the point P. Join OP. Then TP touches a circle of which OP is a radius, therefore OPT is a right angle. Join OT. Then TP is the base of a right-angled triangle of which the height OP and the hypotenuse OT are constant.

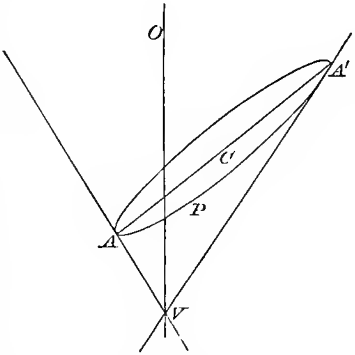
Therefore all the lines drawn from T to touch the sphere are equal. Also the angle OTP is constant, and the equal tangents will therefore generate a cone of which T is the vertex and OT the axis. Also the points common to the sphere and cone lie on a circle whose plane is at right angles to OT.

CHAPTER III.

On the Ellipse.

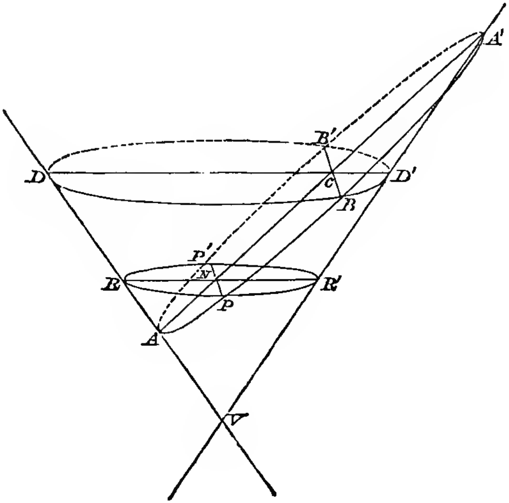
1. **Definition.** Let AVA' be a cone whose axis VO is in the plane of the paper, and VA, VA' its generating lines in that plane. And let the cone be cut by a plane APA' at right angles to the plane of the paper and intersecting it in the line AA' such that A, A' are points in the generating line on opposite sides of the axis VO; then the section APA' is called an *Ellipse.*

It is manifest that the ellipse will be a closed curve, and divided into equal parts by the line AA': every line in the plane of the section drawn at right angles to AA' will meet the curve in two points on opposite sides of AA', and at equal distances from it.



AA' is called the axis major of the ellipse, and a line through C its middle point in the plane of the curve at right angles to AA' limited by the surface of the cone is called the axis minor.

2. Every ellipse may be considered as the projection of a circle whose diameter equals the axis major of the ellipse.



Let DBD', RPR' be two circular sections of the cone made by planes at right angles to the axis through C, N, the middle and any other point of AA', and cutting the plane of the curve in BCB', PNP' at right angles to AA'.

$$\text{Then } NP^2 = RN.NR' \text{ and } BC^2 = DC.CD'.$$

$$\text{Also } RN : DC :: AN : AC \text{ and } NR' : CD' :: NA' : A'C;$$

$$\therefore RN.NR' : DC.CD' :: AN.NA' : AC^2,$$

$$\therefore NP^2 : BC^2 :: AN.NA' : AC^2.$$

Now if NQ be the ordinate of a circle drawn on AA' as diameter, we shall have $NQ^2 = AN.NA'$,

and $$NP^2 : BC^2 :: NQ^2 : AC^2,$$

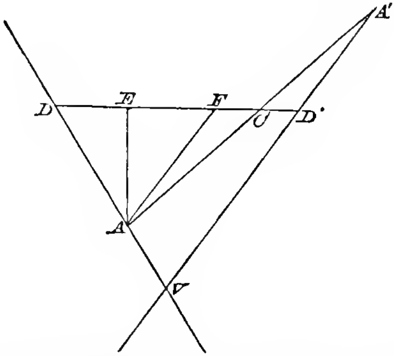
or $$NP : BC :: NQ : AC,$$

$$NP : NQ :: BC : AC;$$

then if a circle, diameter AA', be inclined to the plane of the paper at such an angle that the maximum ratio of diminution of the planes be $AC : BC$, every ordinate of the circle will be diminished by projection in this ratio, and will therefore after projection equal the corresponding ordinate of the ellipse, and the circle whose diameter is AA' will project into an ellipse whose axes equal AA', BB'.

3. We must now prove that in every case the diameter BB' of the ellipse at right angles to the plane of the paper is $< AA'$ in this plane.

$BC^2 = DC . CD'$: we have to shew that $DC . CD' < AC^2$.



Draw AE perpendicular to DD', and AF parallel to $D'A'$ so that CF will $= CD'$.

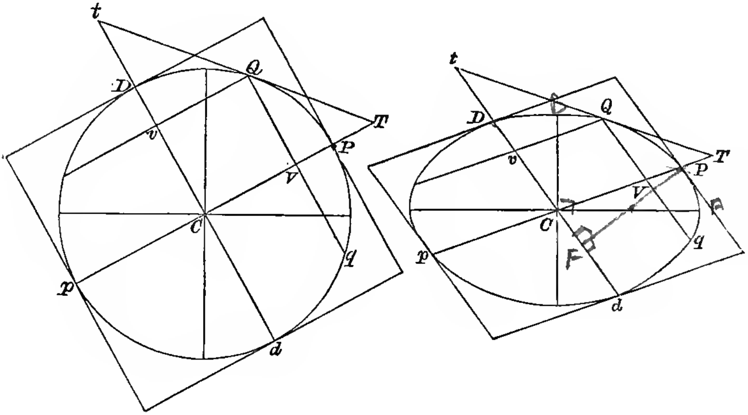
Then $CA^2 = CE^2 + AE^2$

$$= DC . CF + EF^2 + AE^2 \text{ (Euclid II. 6)}$$
$$= DC . CD' + AF^2 \text{ (or } AD^2\text{)},$$

thus $DC . CD'$ is always $< CA^2$, and the ratio $BC : AC$ one of less inequality.

We may now trace some of the properties of the circle in the ellipse by means of the principles of projection already proved.

4. In the circle if two diameters are at right angles to one another each bisects all chords parallel to the other. But by Chap. I. § 4, the projection of a point which bisects a line bisects the projection of the line; therefore chords of the circle which are bisected by a diameter will project into chords of an ellipse bisected by the projection of the diameter. Hence diameters of the circle at right angles to one another project into diameters of the ellipse which bisect each the chords parallel to the other. These diameters of the ellipse are said to be conjugate on account of the relation between them being mutual.



Also the tangents to a circle at the extremities of a diameter are at right angles to the diameter and parallel to the chords bisected by it: and the parallelism of straight lines is not destroyed by projection; hence the tangents at the two extremities of a diameter of an ellipse are parallel to the chords bisected by it and to the conjugate diameter.

Those angles at the centre contained by conjugate diameters in which the axis major lies will in all cases be less than a right angle.

The accompanying figure shews how diameters of the circle at right angles to one another project into conjugate diameters of the ellipse.

5. *Supplemental Chords.* The angle in a semicircle is a right angle in whatever point of the circumference the chords containing it meet: therefore when projected these chords are parallel to conjugate diameters of the ellipse: such chords when projected are called *supplemental chords;* they form two sides of a triangle the base of which is a diameter, and its vertex on the circumference of the ellipse. Hence supplemental chords are parallel to conjugate diameters.

6. If QV be a semi-chord of a circle at right angles to a diameter PCp, and CD the radius parallel to QV, then $QV^2 = PV.Vp$; or since CP, CD are equal,

$$QV^2 : CD^2 :: PV.Vp : CP^2.$$

These ratios are compounded of the ratios of parallel lines, and are therefore unaltered by projection, therefore also in the ellipse

$$QV^2 : CD^2 :: PV.Vp : CP^2.$$

Definition. The half of a chord bisected by a diameter (as QV by PCp) is called an *ordinate* to that diameter.

7. If QT, the tangent to the circle at Q, meet CP produced in T, $CV.CT = CQ^2$ or $= CP^2$; or CP is a mean proportional between CV and CT. The ratios of these lines are not altered by projection; therefore also in the ellipse

$$CV.CT = CP^2.$$

As a particular case of this proposition, if the tangent at P meets the axis major in T, and PN be the ordinate,

$$CN.CT = CA^2.$$

8. The tangents to a circle at the extremities of any chord meet in the diameter (produced) which bisects the chord: and the tangent at the extremity of the diameter when terminated by the pair of tangents is bisected at its point of contact. These properties are not affected by projection, and therefore hold true in the ellipse.

COR. Hence to draw tangents to an ellipse from any point T, join CT and let it cut the curve in P; draw the tangent at P by the latter part of the last Art. In CP take a point V such that CV is a third proportional to CT, CP. Through V draw an ordinate QVq parallel to the tangent at P: join TQ, Tq, then by the proposition of the last Art. these will be the tangents required.

9. The above propositions might equally have been made to take the form, if Qv be the ordinate to CD, and the tangent at Q meet CD in t,

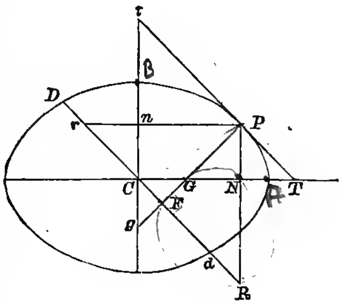
$$Qv^2 : CP^2 :: Dv . vd : CD^2,$$

and $$Cv . Ct = CD^2.$$

As a particular case of the latter, if Pn be the ordinate from P to the axis minor, and the tangent at P meet the axis minor in t, $$Cn . Ct = CB^2.$$

10. $PF . PG = BC^2$, $PF . Pg = AC^2$.

Let the normal at P (*the line through P at right angles to the tangent*) meet the axes and DCd the diameter conjugate to CP in G, g and F; and let the ordinates PN, Pn produced meet Dd in R, r. Then in the quadrilateral $GFRN$, the angles at F and N are right angles; and the quadrilateral may be inscribed in a circle.

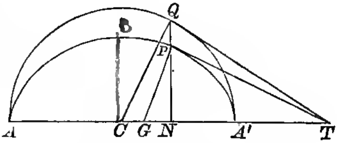


Hence $$PG \, . \, PF = PN \, . \, PR = Cn \, . \, Ct = BC^2.$$

Similarly a circle on rg as diameter will pass through n and F, and $PF \, . \, Pg = Pn \, . \, Pr = CN \, . \, CT = AC^2$.

11. $NG : NC :: BC^2 : AC^2$.

If we draw a circle on the axis major of the ellipse as diameter it is called the auxiliary circle, and will be equal to that which projects into the ellipse; and if the projected circle were made to revolve about its diameter AA', it would come to coincide with the auxiliary circle: and the point which, when the circle was projected, projected into P will now be found in the ordinate NP produced, and if this point be called Q we shall have $NP : NQ :: BC : AC$. Also the tangent to the circle which projects into the tangent PT passes through T, and when made to revolve about AA' will come into the position QT: hence the tangent to the auxiliary circle at Q will meet AA' produced in the same point as the tangent to the ellipse at P.

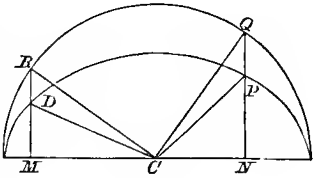


Hence we shall have

$$NG \, . \, NT = NP^2 \text{ and } NC \, . \, NT = NQ^2;$$

$$\therefore NG : NC :: NP^2 : NQ^2 :: BC^2 : AC^2.$$

12. By the reasoning of the last article we see that if CP, CD be conjugate semi-diameters, Q, R the points in which the ordinates NP, MD produced meet the auxiliary circle, QCR will be a right angle.



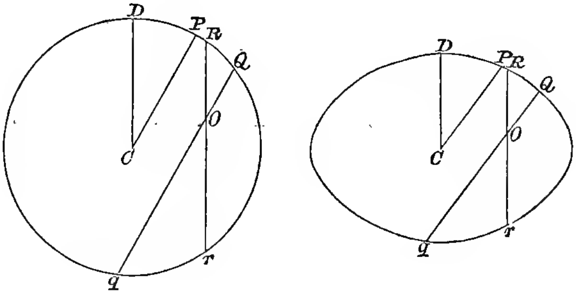
Hence the triangles CQN, RCM will be equal in all respects. Hence

$$CM = QN, \text{ and } CN^2 + CM^2 = CN^2 + QN^2 = CQ^2 = AC^2.$$

Also $QN = CM$, $RM = CN$; $\therefore QN^2 + RM^2 = AC^2$, and $PN^2 + DM^2 : QN^2 + RM^2 :: BC^2 : AC^2$; $\therefore PN^2 + DM^2 = BC^2$, hence $CP^2 + CD^2 = CN^2 + CM^2 + PN^2 + DM^2 = AC^2 + BC^2$.

13. In the circle if the chords Qq, Rr intersect in O, we have $QO \cdot Oq = RO \cdot Or$; and if we draw radii CP, CD parallel to the chords we have

$$QO \cdot Oq : CP^2 :: RO \cdot Or : CD^2.$$



These ratios are not affected by projection, and the same proportion holds in the ellipse: it is independent of the position of O, but depends only on the direction in which the chords are drawn. It is equally true if O be a point exterior to the ellipse. And if tangents be drawn to an ellipse from any point, their lengths will be proportional to the parallel semi-diameters.

Hence if a circle intersect an ellipse in four points and the common chords be drawn, it will easily appear that the semi-diameters to which these chords are parallel must be equal, and therefore they and the chords parallel to them equally inclined to the axes.

14. The area of the parallelogram circumscribing an ellipse and touching it at the extremities of conjugate diameters $= 4AC \cdot BC$.

We have seen that the tangents at the extremities of any diameter are parallel, and if these be drawn at the extremities of two conjugate diameters they will form a parallelogram which is the projection of a square circumscribing the circle whose diameter $= 2AC$, and its area therefore $= 4AC^2$. But by projection this area is diminished in the ratio of $AC : BC$. Hence the area of the parallelogram circumscribing the ellipse having its sides parallel to any pair of conjugate diameters $= 4AC \cdot BC$.

Cor. If PF be the perpendicular from P on the diameter DCd conjugate to CP, then $PF \cdot CD = AC \cdot BC$.

Examples.

1. If an ellipse be projected on a plane intersecting its plane in BC, and having with its plane the maximum ratio of diminution $AC : BC$, it will project into a circle on BC as diameter.

2. If from any point P of an ellipse PQ be drawn parallel to the minor axis to meet the auxiliary circle in Q, and PR parallel to the major axis to meet the circle on the minor axis in R, QR produced will pass through the common centre.

3. A rectilinear figure circumscribing an ellipse and having its sides bisected at the points of contact is the projection of one regular polygon, and will project into another.

4. All such rectilinear figures of a given number of sides are equal in area however placed about the curve.

5. Two triangles circumscribing an ellipse have their sides bisected at the points of contact; the triangular corners cut off from the two triangles, from each by the sides of the other, are equal.

6. A rectilinear figure inscribed in an ellipse has one of its sides bisected at V, and CV produced meets the curve in P; if $CV : CP$ is a constant ratio the same for all the sides, the figure is the projection of one regular polygon and will project into another.

7. If PCP', DCD' are conjugate diameters, then PD, PD' are proportional to the diameters parallel to them.

8. All triangles circumscribing an ellipse which have the line joining each angular point with the point of contact of the opposite side passing through the centre, are equal and are less than any other that has not this property.

9. Any two diameters bisecting supplemental chords are conjugate.

10. Diameters which coincide with the diagonals of the parallelogram on the axes are equal and conjugate.

11. A straight line parallel to the axis major joins the extremities of equal conjugate diameters and intersects the circle on the axis minor in two points. The lines from the centre to these points are at right angles to one another.

12. The length of the line joining the extremities of equal conjugate diameters is independent of the magnitude of that axis to which it is at right angles.

13. The diagonals of any parallelogram formed by tangents at the extremities of conjugate diameters coincide with conjugate diameters.

14. The angular points of these parallelograms lie on one ellipse concentric, similar and similarly situated with the original.

15. In a circle, if the tangents be drawn at the extremities of a chord which passes through a fixed point within the circle, the point of intersection of the pairs of tangents lies always on a straight line. This is called the polar of the point which is called the pole. If Q be the pole, the polar is at right angles to CQ produced, and if CQ produced meets the polar in T, then $CQ \cdot CT =$ square on the radius. Prove this, and transfer the property to the ellipse, shewing that if CQT meets the ellipse in P, $CQ \cdot CT = CP^2$.

16. If in each of two ellipses the two axes are in the same ratio to one another, shew that radii which are equally inclined to either (say the major) axis in both bear to one another a constant ratio, i.e. the curves are similar.

17. Parallel elliptic sections of the same cone are similar.

18. If in Ex. 15, Q lies always upon an ellipse similar and similarly situated to the original ellipse, intersecting it and having its tangents at the points of intersection meeting in C, then T will be found upon the part of the second ellipse which lies without the curve.

19. The rectangle of the segments of any tangent intercepted between two parallel tangents made by the point of contact = square on the parallel semi-diameter.

20. The rectangle of the intercepts on parallel tangents made by any other tangent = square on the semi-diameter parallel to them.

21. If any tangent meet any two conjugate diameters, the rectangle under its segments = square on the parallel semi-diameter.

22. If from the extremity of each of any two semi-diameters ordinates be drawn to the other, the two triangles so formed will be equal in area.

23. Or if tangents be drawn from the extremity of each to meet the other produced, the two triangles so formed will be equal in area.

24. If CA, CB be the semi-axes of an ellipse and the rectangle $BCAD$ is completed: shew that if an ellipse similar and similarly situated to the given ellipse be described about it, and any chord $DPQR$ be drawn cutting the first ellipse in P, R, and the second in Q, $PQ = QR$.

25. Inscribe in a triangle an ellipse with axes parallel to given lines, and in a given ratio.

26. If the elliptic area between the two radii CP, CQ is invariable, prove that the area between the chord PQ and the curve will be so also.

27. PD joins the extremities of conjugate diameters, and CIJ makes a constant angle with PD, meeting it in I, and the parallel tangent in J. Shew that I and J trace similar curves.

28. If P be any point of an ellipse, and AP, $A'P$ produced meet the tangents at A', A in R and S, the tangent at P will bisect AS and $A'R$.

29. If from an external point O, two straight lines OAP, OQA' be drawn through the vertices of an ellipse $APA'Q$: if QA, $A'P$ intersect in R, OR is at right angles to the axis major.

30. If TP, TP' be tangents to an ellipse, and PCp be the diameter through P, then $P'p$ is parallel to CT.

31. If TP, TP' be tangents to an ellipse from an external point T, TR the diagonal of the parallelogram on TP, TP' and R be on the ellipse, then T will lie on an ellipse similar and similarly situated to the former.

32. If from any point T exterior to an ellipse, a line be drawn parallel to either axis to meet the curve the first or second

time in Q, the line bisecting TQ at right angles and that bisecting the tangents from T will meet on the tangent at Q.

33. Find the centre of a given ellipse.

34. Find the axes of a given ellipse.

35. In a given ellipse find the diameter conjugate to a given diameter.

36. If two points of a rod be constrained to move in two fixed lines which intersect at right angles, every other point of the rod will describe an ellipse.

37. If from any point P of an ellipse PQ be drawn to the axis major equal to BC, then if PQ produced either way meets the axis minor in R, $PR = AC$.

38. If PCP', DCD' are conjugate diameters, and PQ is drawn parallel to the axis major to meet the curve in Q; prove that DQ is parallel to two of the lines joining extremities of the axes of the curve.

39. If NP produced meets the auxiliary circle in Q, prove that GP, CQ produced meet on a circle whose diameter = sum of the axes of the curve.

40. Two ellipses have their axes equal each to each and in the same plane, also their centres coincident, draw the common tangents.

41. If CA, CB be any conjugate diameters of an ellipse and CB be produced to any point B', and an ellipse is described on CA, CB' as conjugate diameters; if the ordinate $P'PN$ be drawn parallel to BC, shew that the tangents to the ellipses at P, P' will intersect at a point lying on CA produced, and that

$$PN : P'N :: BC : B'C.$$

42. If two ellipses with major axes parallel or at right angles intersect in four points, the opposite sides of the quadrilateral formed by joining the four points will be equally inclined to either axis of either curve.

43. If any system of diameters of an ellipse of a given even number divide it into equal sectors, the sum of the squares on the diameters is the same whatever their directions.

44. The same when the number is odd or even.

45. Prove that $PG \, . \, Pg = CD^2$.

46. If CR be the perpendicular from the centre on the tangent at P, and BR, AD be joined; prove that the triangles ACD, RCB are similar.

CHAPTER IV.

Focal Properties of the Ellipse.

1. We have now to shew that there are two points within the ellipse which, regarding the curve in its relation to the circle, are as it were a divided centre, and also two lines exterior to it which bear a remarkable relation to these points and the curve.

2. Let us premise the following propositions:

If the circle inscribed in the triangle VAA' touches the sides at K, S, L; and the circle that touches AA' and VA, VA' produced touches them at H, K', L', then $KK' = AA'$, and $AS = A'H$.

For the perimeter of triangle VAA'

$$= VA + AH + HA' + A'V$$
$$= VA + AK' + L'A' + VA' = VK' + VL' = 2VK'$$
$$= 2VK + 2KK':$$

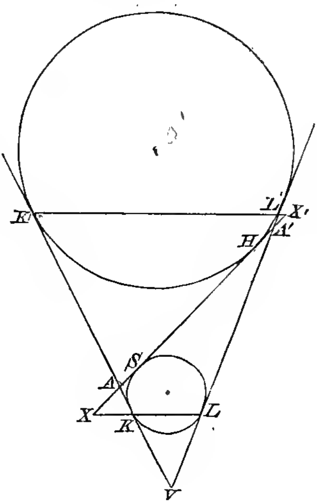
also

$$= VK + KA + AA' + A'L + LV$$
$$= VK + AS + AA' + A'S + LV = 2VK + 2AA',$$
$$\therefore KK' = AA'.$$

Also $\quad KK' = KA + AK' = AS + AH,$

$$\therefore AA' = AH + AS: \text{ but } AA' = AH + A'H,$$
$$\therefore AS = A'H.$$

Also if we produce AA' both ways to meet LK, $K'L'$ produced in X, X', AX will $=A'X'$.

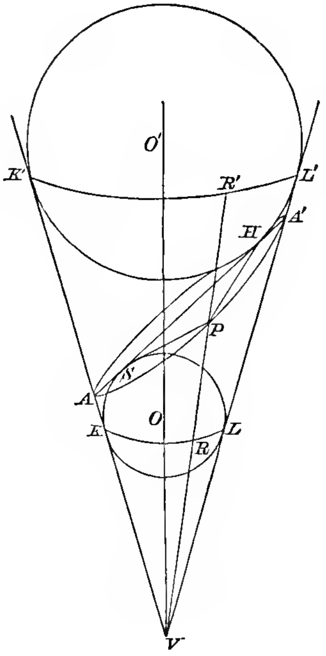


For AK, $A'L'$ being equal to AS, $A'H$ are equal, and being equally inclined to the axis of the cone will have equal projections on it. And the projections of AX, $A'X'$ will equal those of AK, $A'L'$, and are therefore equal: therefore AX, $A'X'$, being parts of the same line with equal projections on the axis of the cone, are equal.

3. Now to return to the construction in Chap. III. § 1. Let AA' be the intersection of the plane of the paper with a plane at right angles to it that cuts the cone VAA' in the ellipse APA'.

Inscribe in the triangle VAA' the circle SKL with centre O on the axis of the cone, and escribe the circle $HK'L'$ with centre O' also on the axis. Then if we make the circles to revolve about their diameters which coincide with the axis of the cone, they will generate spheres which will touch the cone in the circles KRL, $K'R'L'$. Every point of each of these circles is equidistant from V, and therefore the distance RR' from one circle to another along a generating line of the cone will be invariable. Also OS, $O'H$ will be at right angles to the cutting plane, which will therefore touch the spheres in S and H.

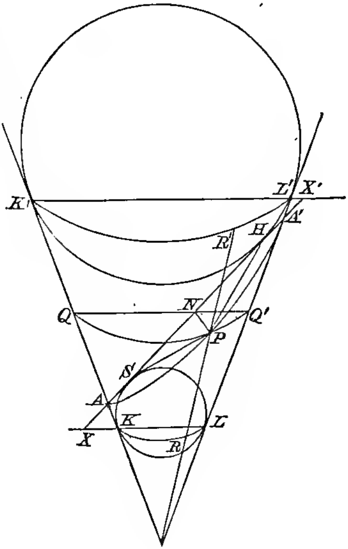
Let P be any point of the elliptic section, draw $VRPR'$ the generating line of the cone through P, and join SP, HP.



Then PS, PR are drawn from P to touch the sphere centre O in S and R, therefore $PS = PR$, and PH, PR' touch the sphere centre O' in H and R', therefore $PH = PR'$. Hence $SP + HP = PR + PR' = RR' = KK' = AA'$.

Hence the sum of the distances of any point of the ellipse from the points S, H within it is invariable and $= AA'$.

Hence we obtain the following construction to enable us to describe an ellipse: Fasten the two ends of a thread to the two points S and H, and let the thread be longer than SH: then stretch it with the point of a pencil, and mark the line which is traced by moving the pencil on all sides of the points so as to keep the thread tightly stretched: the curve so traced will be an ellipse. Viewing the ellipse in its relation to the circle, S and H may be considered as a divided centre, the sum of the distances of all points in the circumference from them being the same.



4. Now let the plane through P, perpendicular to the axis of the cone, cut the cone in the circle QPQ', and the plane of the ellipse in NP at right angles to AA'.

Then we shall have

$$SP = PR = QK,$$

and the triangles QAN, KAX are similar;

$$\therefore QK : NX :: AK : AX$$
$$:: AS : AX.$$

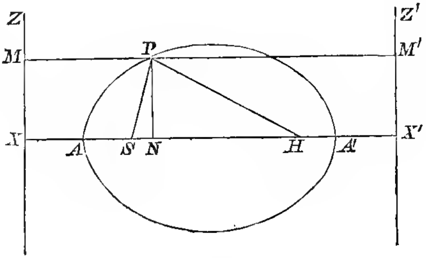
Hence whenever P is situated on the ellipse, $SP : NX$ is a constant ratio $= AS : AX$.

And similarly, joining HP,

$$HP\,(= PR' = Q'L') : NX' :: A'H : A'X'.$$

The ratios $AS : AX$, $A'H : A'X'$ are equal, and since $A'L' < A'X'$, each is a ratio of lesser inequality: either of them is called the *eccentricity*.

So then if we draw the elliptic section in the plane of the paper, and draw through X, X' lines XZ, $X'Z'$ at right

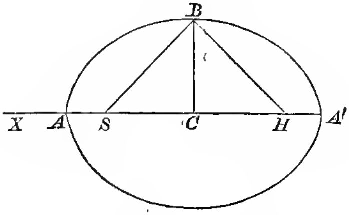


angles to XX', and PM, PM' perpendiculars to these lines; we have for any point P of the ellipse

$$SP : NX \text{ (or } PM) :: AS : AX,$$
$$HP : NX' \text{ (or } PM') :: A'H : A'X'.$$

S and H are called the *foci* of the ellipse, XZ, $X'Z'$ the *directrices.*

5. The position of the foci and directrices is determined



by the following relations: $CS^2 + CB^2 = CA^2$:

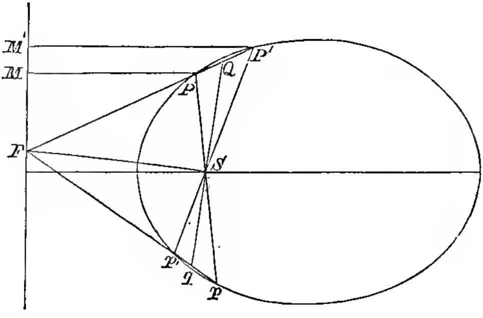
For joining the foci with the extremity of the axis minor, $SB = HB = \frac{1}{2}(SB + HB) = AC$, and $SC^2 + BC^2 = SB^2 = AC^2$.

Also CX is a third proportional to CS and CA.

For first, $SB : CX$, i.e. $CA : CX :: SA : AX$.

Hence $$CA : CX :: SA : AX,$$
$$CA : SA :: CX : AX,$$
$$CA - SA : SA :: CX - AX : AX,$$
or $$CS : SA :: CA : AX,$$
$$\therefore \quad CS : CA :: SA : AX,$$
$$:: CA : CX,$$
or CX is a third proportional to CS and CA.

6. Properties of the secant and tangent.



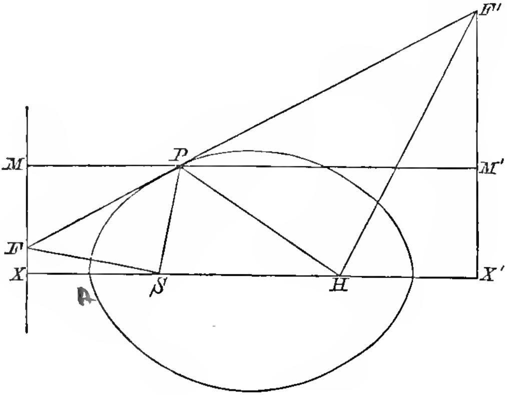
Let PSp, $P'Sp'$ be two focal chords. Let the secant $P'P$ cut the directrix in F, join SF. Let QSq bisect the vertical opposite angles PSP', pSp': PM, $P'M'$ perpendicular to the directrix.

Then $SP : SP' :: PM : P'M' :: PF : P'F$,

therefore SF bisects the angle PSp' (Euclid VI. A) and is perpendicular to QSq; hence also the secant pp' will pass through F.

Now let the secant FPP' revolve about F, so that P, P' will approach one another; SQ still bisecting the angle PSP' will be perpendicular to SF and therefore constant in position; and P, P' will finally coincide with Q which will then be a point on the curve, and the secant will then become a tangent to the curve at that point. Similarly if the secant $Fp'p$ turn about F, it will in its limiting position touch the curve in Sq produced. Hence the tangents at the extremity of a focal chord intersect on the directrix: and *the part of any tangent between the curve and the directrix subtends a right angle at the focus.*

7. The focal distances make equal angles with the tangent at any point.



Let the tangent at P meet the directrices in F, F', join SF, HF'; PSF, PHF' will be right angles.

Also the triangles MPF, $M'PF'$ are similar,

and $$SP : PM :: AS : AX :: HP : PM';$$

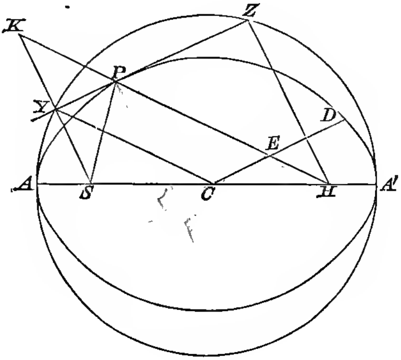
$$\therefore\ SP : PH :: PM : PM' :: PF : PF',$$

or $$SP : PF :: HP : PF'.$$

Hence (Euclid VI. 7) SPF, HPF' are similar triangles, and the angles SPF, HPF' are equal.

COR. The tangents at the extremities of the axes are at right angles to them.

8. The feet of the perpendiculars from the foci on any tangent lie on the auxiliary circle.



Let SY the perpendicular from S on the tangent at P meet HP produced in K. Join SP, CY. Then in the right-angled triangles SYP, KYP, PY is common and the angle $SPY =$ the angle HPZ (by the last proposition)

$=$ the vertical opposite angle KPY.

Hence $SY = KY$, and $SP = KP$:

$$\therefore\ KH = KP + PH = SP + PH = 2AC.$$

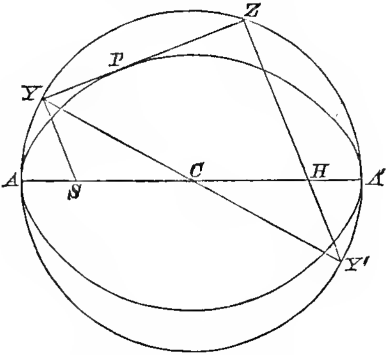
And C, Y being the middle points of SH, SK, CY is parallel to HK and half of it and therefore $= AC$.

Hence Y lies on the circle on AA' as diameter (the auxiliary circle). And similarly Z the foot of the perpendicular from H.

Cor. 1. CY being parallel to HP and bisecting SH, also bisects SP. Hence SYP being a right angle, the circle on SP as diameter passes through Y, and has its centre on CY; hence it touches the auxiliary circle at Y.

Cor. 2. If CD be drawn parallel to the tangent at P and therefore conjugate to CP, and intersect PH in E, then $CEPY$ is a parallelogram, and $PE = CY = AC$.

9. If SY, HZ are perpendiculars on the tangent at P from the foci S, H; $SY\,.\,HZ = BC^2$.



Since YZH is a right angle, YC produced will meet ZH produced on the auxiliary circle (at Y'). The triangles SCY, HCY' are equal in all respects, and

$$SY\,.\,HZ = HY'\,.\,HZ = AH\,.\,A'H = A'C^2 - CH^2 = BC^2.$$

10. $SP \, . \, HP = CD^2$.

If we draw PF in fig. § 8 perpendicular to DC (produced if necessary) we shall have the triangle PEF similar to HPZ and SPY; and $PE = AC$: hence

$$SP : SY :: HP : HZ :: PE : PF :: AC : PF$$

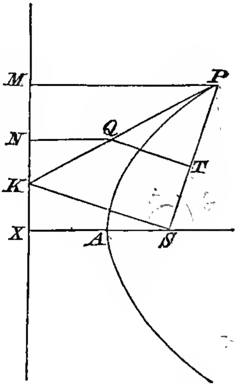
$$:: CD : BC, \quad \because \; PF.CD = AC.BC;$$

$$\therefore \; SP \, . \, HP : SY \, . \, HZ :: CD^2 : BC^2, \text{ and } SY \, . \, HZ = BC^2,$$

$$\therefore \; SP \, . \, HP = CD^2.$$

11. If from any point Q of the tangent PK perpendiculars QN, QT be drawn to the directrix and SP, then

$$ST : QN :: AS : AX.$$



For QT is parallel to KS; and if we draw PM perpendicular to the directrix,

$$ST : SP :: QK : PK :: QN : PM,$$

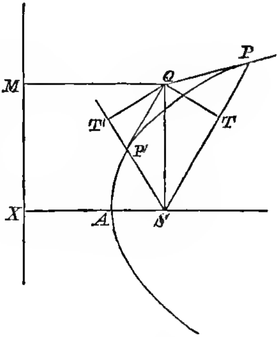
$$\therefore \; ST : QN :: SP : PM :: AS : AX.$$

The student of analytical geometry will see in this proposition the basis of the polar equation to the tangent in terms of the angle PSA (α), viz. $\rho = \dfrac{a(1-e^2)}{\cos(\alpha-\theta) + e\cos\theta}$.

The proposition is due to Professor Adams, and the property is equally true of all the Conic Sections.

12. Hence if QP, QP' be the tangents to an ellipse from an exterior point Q, QP, QP' subtend equal angles at S.

For if we draw QT, QT', QM the perpendiculars on SP, SP' and the directrix, we shall have ST, ST' in the same proportion to QM and therefore equal.

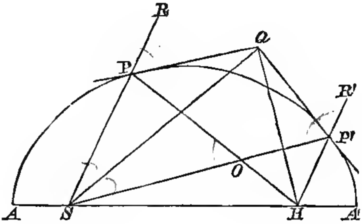


Hence the right-angled triangles QST, QST' are equal in all respects, and the angles QSP, QSP' equal.

If Q lie beyond the directrix, T, T' will lie in PS, $P'S$ produced, and the angles QSP, QSP' will be proved equal by proving their supplements equal.

13. If QP, QP' be the tangents to the ellipse from Q, the angles SQP, HQP' are equal.

Produce SP, HP' to R, R'; and let HP, SP' intersect in O.



Then QP, QS bisect the angles HPR, $P'SP$ respectively, also OPR = sum of the interior opposite angles OSP, SOP, and QPR = sum of QSP, SQP.

$\therefore$ sum of OSP, SOP = twice the sum of QSP, SQP,

of which OSP = twice QSP,

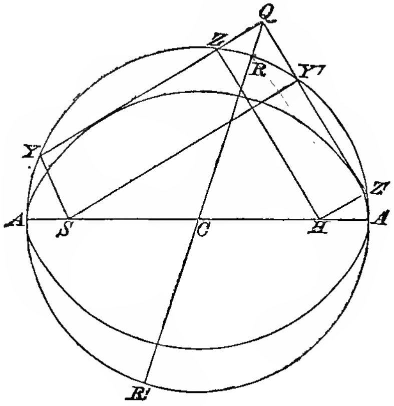
$\therefore$ angle SOP = twice SQP.

Similarly the angle HOP' = twice HQP'.

And the angle $SOP = HOP'$, $\therefore$ $SQP = HQP'$.

14. Any two tangents at right angles to one another intersect on a circle whose centre is C and square on the radius $= CA^2 + CB^2$.

Let any two tangents at right angles to one another cut



the auxiliary circle in Y, Z; Y', Z'. Draw SY, SY'; HZ, HZ'. Let QC cut the circle in R, R'.

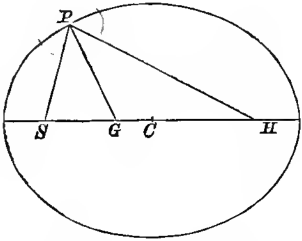
Then SQ, HQ will be rectangles, with opposite sides equal.

Hence $QR \cdot QR' = QY' \cdot QZ' = SY \cdot HZ = BC^2$,

$$\therefore CQ^2 = CR^2 + QR \cdot QR'$$
$$= CA^2 + BC^2.$$

15. $$SG : SP :: SA : AX.$$

The tangent at P makes equal angles with SP, PH.



Hence the normal PG bisects the angle SPH. Hence (Euclid VI. 3)

$$SG : SP :: HG : HP :: SG + HG : SP + PH$$
$$:: 2SC : 2AC :: SC : AC :: SA : AX.$$

EXAMPLES.

1. A series of ellipses pass through a point and have a common focus and their axes major are equal: shew that the other focus always lies on the circumference of a fixed circle.

2. Under the same circumstances the centre also describes a circle.

3. Under the same circumstances, what is the greatest excentricity the ellipse can have, and what does it then become?

4. Given three points of an ellipse and one focus, shew how to find the corresponding directrix.

5. The lines joining the extremities of a focal chord with the vertices have their points of intersection on the corresponding directrix.

6. The part of a directrix intercepted between the lines joining any point of the ellipse with the vertices subtends a right angle at the corresponding focus.

7. Two circles have their centres fixed and the sum of their radii constant: find the locus of the centre of a circle of constant radius that touches them both.

8. A circle through Y, Z, the feet of the perpendiculars on the tangent from the foci touches the axis major in Q, and has its centre O in the tangent: shew that $OQ=BC$.

9. If the tangent at a point P, the foot of whose ordinate is at N, intersects the major axis produced in T: prove that $TN \,.\, TC = TA \,.\, TA'$.

10. A circle through Y, Z, and N the foot of the ordinate at the point of contact, will also pass through the centre of the ellipse.

11. The focal distances of two points where the tangents are parallel form a parallelogram.

12. SY is perpendicular at the tangent at P, HY' parallel to PS meets YS produced in Y', prove $HY'=2AC$.

13. If CF, perpendicular to the tangent at P intersect HP in F, $HF=AC$.

14. The locus of the intersections of the perpendicular from

the centre on the tangent with the focal distances is two circles with radius AC, and centres at the foci.

15. If CD cut SP and HP in E and E', prove $SE = HE'$ and the circle round SEC=that round $HE'C$.

16. If from the centre of an ellipse lines be drawn parallel and perpendicular to a tangent at any point, they inclose a part of one of the focal distances of that point which equals the other focal distance.

17. If SP, HP are at right angles to one another

$$SP \,.\, HP = 2BC^2.$$

18. Given the two foci and one tangent of an ellipse to draw the directrices.

19. If from any point T of a tangent at P, TQ be drawn at right angles to SP produced if necessary and TRN at right angles to the axis major cuts the curve in R, then $SQ = SR$.

20. The tangent at the extremity of the latus rectum meets the axis major at the foot of the directrix.

21. The circle that passes through L, L' the extremities of the latus rectum through S and touches the nearer directrix is touched by LH.

22. Shew that lines drawn from a focus to points on the ellipse at equal distances from the extremities of the axis minor are equally inclined to the tangents at the points.

23. A ship sails over an elliptic path having its middle point at B, shew that she changes her direction as much as an observer at the focus does the direction of his telescope in watching her.

24. Given a focal chord and the tangents at its extremity, find the second focus.

25. Given a focal chord and the second focus, find that which lies on the chord.

26. Given PO, QO tangents to an ellipse at the points P, Q; POQ is less than a right angle and it is known that one focus lies in PQ, find the other and the directrices.

27. If PN be an ordinate, the angle $PNY =$ the angle PSY.

28. If PN be an ordinate, PN bisects the angle YNZ.

29. If a line through X the foot of the directrix cut the ellipse in P, p, SP, Sp are equally inclined to either axis. If PSQ be a focal chord PS, SQ subtend equal angles at X.

30. The circle on PG as diameter cuts SP, HP in K and L, shew that KL is perpendicular to PG and bisected by it.

31. Prove $SG : HG :: SY : HZ$.

32. SZ and HY each bisect the normal PG.

33. If DR be the ordinate at D, and CD conjugate to CP, the triangles PGN, DRC are similar, and in any ellipse PG is proportional to CD.

34. If QPN be the common ordinate to the auxiliary circle and the ellipse, and the tangents at Q and P meet on the axis major in T, prove $TQ : TP :: BC : PG$.

35. If O, O' be the centres of the circles inscribed in the triangle SPH, and escribed on its side SH, then $PO \, . \, PO' = CD^2$.

36. If ET be the side of a parallelogram whose sides touch an ellipse at the extremities of conjugate diameters, and E, T when joined each with the two foci have their joining lines meeting in O and O'; then shew that O, S, O', H lie on the same circle; and that the sum of the angles subtended by SH at E, T and $O' =$ that it subtends at O.

37. A diameter CP produced intersects the directrix in V, prove that VS is at right angles to the diameter conjugate to CP.

38. Given the centre and directrix of an ellipse, also the directions of a pair of conjugate diameters, determine the position of the foci.

39. If tangents PT, QT meet on the auxiliary circle, prove that SP is parallel to HQ.

40. If a parallelogram circumscribes an ellipse, the lines joining the points of contact also form a parallelogram.

41. A parallelogram is described about an ellipse having two of its corners on directrices, prove that the other two will lie on the auxiliary circle.

42. If a line be drawn through a focus perpendicular to two parallel tangents, the rectangle of its segments made by the focus $= BC^2$.

43. A line is drawn through the focus of an ellipse perpendicular to a pair of parallel tangents: on this line as diameter a circle is described: prove that the chord of this circle parallel to the tangents such as when produced passes through the other focus $= BB'$.

44. If a quadrilateral be circumscribed about an ellipse, the angle subtended at either focus by opposite sides are supplementary.

45. Circles are described on SP, HP as diameters, and chords of these circles YI, ZJ are drawn at right angles to the axis major, prove that SI, HJ produced intersect on the axis minor.

46. The points in which the tangents at the vertices are intersected by any tangent are joined each with a focus; shew that these lines intersect in the normal.

47. The tangents at the extremities of the latera recta on the same side of the axis major intersect on the circumference of the circle through the foci and the points of contact.

48. If tangents PT, QT meet in T and one of them QT is produced to any point Q', prove that the angle PTQ' is a mean between PSQ and PHQ.

49. An endless string of greater length than the circumference of an ellipse which is laid on a sheet of paper is made to pass round it and stretched tight by a pencil: prove that the point of the pencil will trace an ellipse having the same foci as the original. (It may be assumed that the normal to the pencil's path at each point will make equal angles with the directions of the string at the point.)

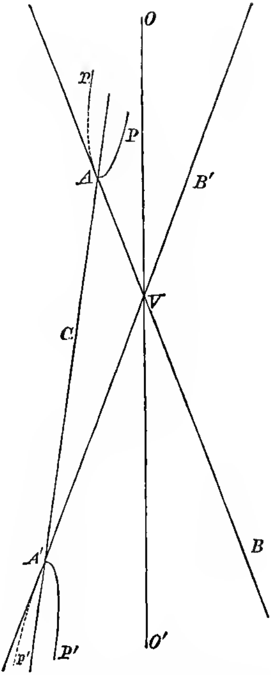
50. If P be the vertex of a triangle whose base AB is bisected in C. Then if $AP \,.\, BP + CP^2$ is a constant quantity, the locus of P is an ellipse.

51. If AD be the portion of the generating line of a cone which contains the vertex of an ellipse cut from it intercepted between the vertex and a line through C at right angles to the axis of the cone; shew that $AD = CS$.

52. To cut an ellipse of given axes from a given cone.

CHAPTER V.

On the Hyperbola.



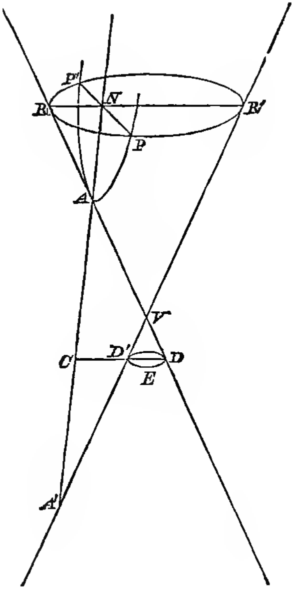
1. DEFINITION. Let AVB', BVA' be the two sheets of a cone, whose axis OVO' is in the plane of the paper and AB $A'B'$ the two positions of the generating line in that plane. Let the cone be cut by a plane APA' perpendicular to the plane of the paper which cuts it in the line AA', such that A, A' are points in the generating line of the cone on the same side of the axis OVO': then if this plane cuts the cone in the lines PAp $P'A'p'$, these lines make up a curve which is called an *Hyperbola*.

It is manifest that the two branches of this curve may be prolonged to any length and cannot intersect. Also they are each divided into two equal parts by the line AA' produced: every line in the plane of the section drawn at right angles to AA' will meet the curve in two points on opposite sides of AA' and at equal distances from it.

AA' is called the *transverse axis* of the hyperbola, and a line through C the middle point of AA', at right angles to AA' in the plane of the section and of a magnitude to be specified in the next article, is called its *conjugate axis*.

2. Every hyperbola may be projected from or into an hyperbola whose axes are equal, called an *equilateral* or *rectangular* hyperbola.

Let DED', RPR' be circular sections of the cone, made by planes at right angles to the axis through C and through



N any point of AA' produced, and let this latter plane cut the plane of the curve in PNP' at right angles to AA'. Then $NP^2 = RN \,.\, NR'$.

Also in the similar triangles NAR, CAD

$$RN : DC :: AN : AC,$$

and in the similar triangles $NA'R'$, $CA'D'$

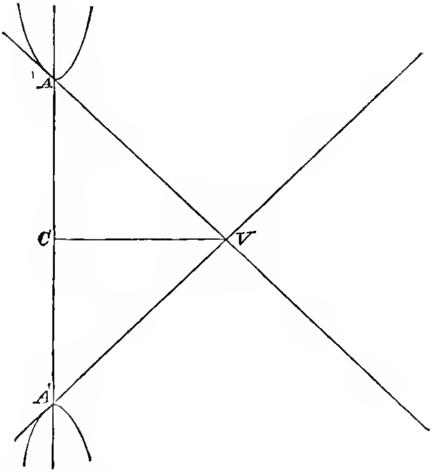
$$\begin{aligned} NR' : CD' &:: A'N : A'C \\ &:: A'N : AC \end{aligned}$$

$$\therefore RN . NR' : CD . CD' :: AN . A'N : AC^2,$$

or $$NP^2 : AN . A'N :: CD . CD' : AC^2.$$

Now if we draw CE from C to touch the circle DED in E, $CE^2 = CD . CD'$: and we may now complete our definition of the conjugate axis of the hyperbola by saying that it is equal in length to CE.

Now let us take a cone whose vertical angle is a right



angle, and let an hyperbola be cut from it by a plane parallel to the axis and such that the transverse axis shall equal AA', then D, D' will merge in V, and CE will $= CV$, will also $= CA$, since the circle with centre C and distance CA or CA' will pass through V, AVA' being a right angle.

Hence the hyperbola will be rectangular and will have each of its axes equal to AA'.

Let the two hyperbolas be now placed in the same plane with their transverse axes coincident, and let the ordinate NP of the first hyperbola produced if necessary meet the rectangular hyperbola in Q. Then we have

$$NP^2 : AN . A'N :: CE^2 : AC^2,$$

and $$NQ^2 : AN : A'N :: AC^2 : AC^2, \text{ or } NQ^2 = AN . A'N :$$

hence $$NP^2 : NQ^2 :: CE^2 : AC,$$

or $$NP : NQ :: CE : AC.$$

We shall shew that CE may be greater or less than CA: if CE be greater than CA the first hyperbola will project into the rectangular one if, the transverse axes remaining coincident, they are placed in planes inclined to one another at the proper angle. If CE be less than CA the rectangular hyperbola will project into the one of unequal axes.

Cor. The proposition proved above

$$NP^2 : AN . A'N :: CE^2 : AC^2$$

might have been equally proved if we had taken the circular section in the other sheet of the cone below A'. Hence we observe that if we take two points in the transverse axis of the curve produced both ways on opposite sides of C and equidistant from it, the ordinates drawn through them will be equal, and the points in which they meet the curve will be equidistant from the centre. Hence the two branches of the curve are equal and every line through the centre joining them is bisected in the centre.

3. It has been shewn that when the cone has a right angle at the vertex, and the cutting plane is parallel to its axis, the two axes of the curve are equal. Also if the cutting plane is parallel to the axis of the cone, $CE = CA >$ or $< CA$ according as angle of the cone is obtuse or acute. Hence it appears that the axes of an hyperbola may be equal or either in excess of the other. It may be shewn that when the angle of the cone is acute the transverse axis is the greater for all sections; when the angle is a right angle the axes are equal when the cutting plane is parallel to the axis of the cone, and the transverse axis the greater for all other sections. When the angle of the cone is obtuse, either axis may be greater than the other, or they may be equal according to the inclination of the cutting plane to the axis of the cone. This will appear if we draw VE at right angles to the axis of the cone to meet AA' in E and turn AA' about E, observing that

$$CD.CD' : AC^2 :: VE^2 : AE.EA',$$

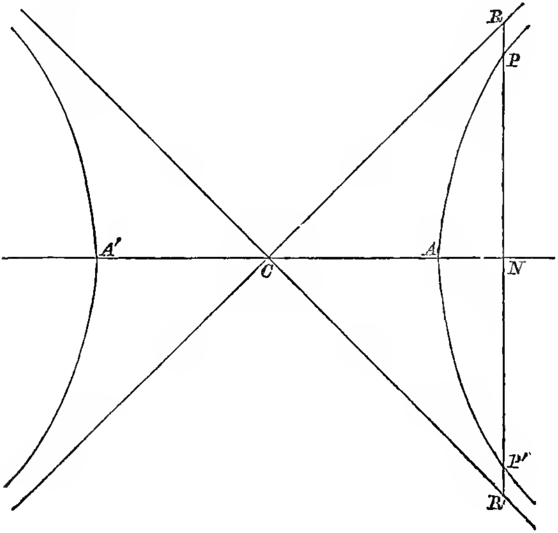
and that $AE.EA'$ is least when AA' is at right angles to EV and ~~diminishes~~ increases the more the further it is turned from that position.

Further, it is evident that hyperbolas cut from any cone will have their axes in the same proportions when the cutting plane is so situated that $CD.CD' : AA'^2$ is a constant ratio. This will be the case when the plane moves parallel to itself: as in that case each of the ratios $CD : AC$ and $CD' : A'C$ remains constant.

If the angle of the cone is made to vary as well as the position of the cutting plane, a curve equal in all respects will be obtained if AA' remains unchanged and the rectangle $CD.CD'$ is also constant.

We may now proceed to investigate the properties of the rectangular hyperbola, and then we can generalise them by projection for the hyperbola of unequal axes.

4. Property of the Asymptotes.



Let PAP' be one branch of the curve, and through C draw CR, CR' each making half a right angle with CA: let $RPNP'R'$ be at right angles to $A'A$ produced. Then $NR = NR' = CN$, and since $A'A$ is bisected in C, and produced to N,

$$\therefore AN \,.\, A'N + AC^2 = CN^2:$$

but $$AN \,.\, A'N = NP^2,$$

$$\therefore NP^2 + AC^2 = CN^2,$$

or $$NP^2 + AC^2 . = NR^2.$$

But RR' is divided into two equal parts in N and unequally in P:

$$\therefore NP^2 + RP.PR' = NR^2.$$

$$\therefore RP.PR' = AC^2.$$

Now the further N is removed from A, the greater does RR' become, and therefore the greater does PR' and the less does PR become, and by increasing AN and therefore PR' we may make PR smaller than any assigned quantity; hence the curve approaches nearer and nearer to the straight line without ever coinciding with it; hence the straight line is said to touch the curve at an infinite distance, and is called an *asymptote*. CR and CR' both approach the curve in the same way, and each approaches the two branches at its opposite extremities. They are both asymptotes and the rectangular hyperbola is so called from the fact, that its asymptotes are at right angles to each other.

5. The intercepts on any straight line between the rectangular hyperbola and its asymptotes are equal.

In the last Art. since $NP = NP'$ and $NR = NR'$, therefore $PR = P'R'$, this proves the proposition for lines at right angles to the transverse axis.

Let any chord Qq be produced to meet the asymptotes in U and u. Let the ordinates at Q, q, meet the asymptotes in R, R', r, r'.

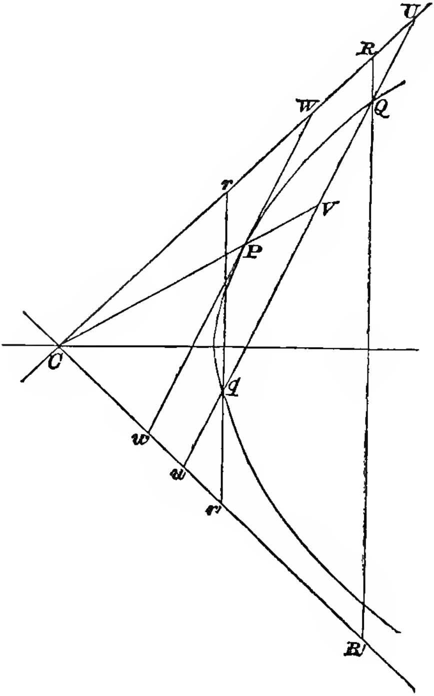
Then $\quad QU : QR :: qU : qr,$

$\quad Qu : QR' :: qu : qr'.$

Hence $\quad QU.Qu : QR.QR' :: qU.qu : qr.qr',$

therefore $QU \cdot Qu : AC^2 :: qU \cdot qu : AC^2$,

therefore $QU \cdot Qu = qU \cdot qu$.



Let V be the middle point of Uu,

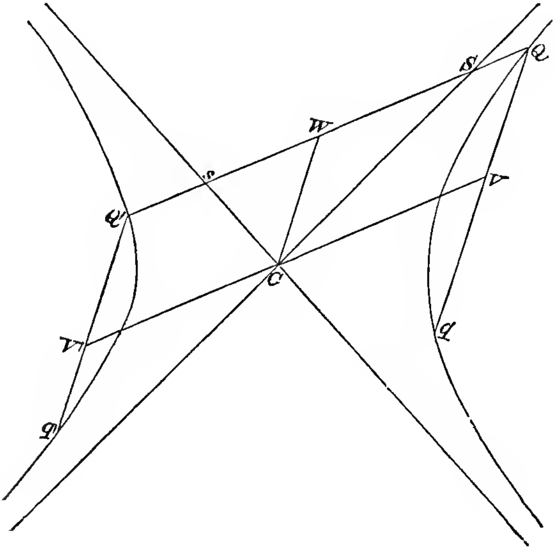
then $QU \cdot Qu + VQ^2 = VU^2 = Vu^2 = qU \cdot qu + Vq^2$,

but $QU \cdot Qu = qU \cdot qu$, therefore $VQ^2 = Vq^2$;

hence V is also the middle point of Qq, and therefore $QU = qu$.

Join CV, cutting the curve in P: all chords of the asymptotes parallel to Qq will be bisected by CV, and therefore also all chords to the curve parallel to the same line. *The locus of the middle points of parallel chords is a straight line through the centre.*

If we further draw CW parallel to Qq and QW parallel to VC, and produce QW to meet the other branch of the curve in Q', W will be the middle point of QQ'.



For the two branches of the curve are in all respects equal and similarly situated with respect to the asymptotes. Therefore if we draw the chord $Q'q'$ parallel to Qq, VC produced will bisect $Q'q'$ (in V' suppose).

Also $Q'V'$ by construction $= WC = QV$: hence $Q'q'$ will equal Qq, and CV' will equal CV: $Q'C$, WV are parallelograms, and therefore

$$Q'W = V'C = VC = QW$$

and W is the bisection of QQ'.

And in like manner all chords between the two branches of the curves parallel to QQ' are bisected by the diameter CW.

Let QQ' meet the asymptotes in S, s: then since the angle at C is a right angle and V is the middle point of Uu, $CV = VU$, and CV, VU are equally inclined to the asymptote; so ∴ are SW, WC being parallel to them:

$$\therefore SW = WC = Ws$$

since the angle at C is a right angle: W bisects Ss, and since it also bisects QQ', ∴ $QS = Q's$. Shewing that every straight line that cuts the curve and the asymptotes, each in two points, and the curve in one branch or both, has the intercepts between the curve and the asymptotes equal.

Cor. Hence as the chord Uu moves parallel to itself, so that Q and q approach each other and finally coincide in P, the chord ultimately becomes a tangent at P and is bisected in P: shewing that *the part of the tangent intercepted between the asymptotes is bisected at the point of contact.*

Each half of the tangent will equal CP.

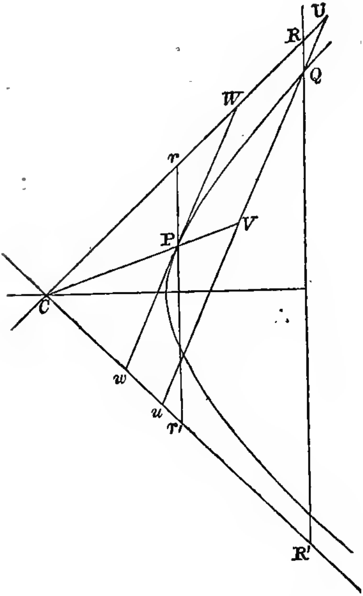
6. $QV^2 + CP^2 = VU^2$.

Let Ww be the chord to the asymptotes which touches the curve at P. Through P draw rPr' the double ordinate to the asymptotes. Then

$$QU : QR :: PW : Pr :: CP : Pr$$

and $$Qu : QR' :: Pw : Pr' :: CP : Pr';$$

$$\therefore QU \,.\, Qu : QR \,.\, QR' :: CP^2 : Pr \,.\, Pr'$$



or $$QU \,.\, Qu : AC^2 :: CP^2 : AC^2,$$

$$\therefore QU \,.\, Qu = CP^2 \text{ and } QV^2 + CP^2 = VU^2.$$

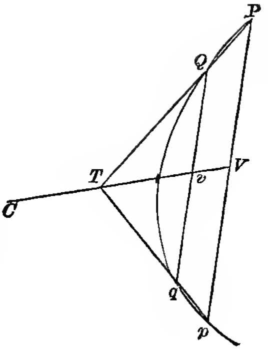
7. The tangents at the extremities of any chord meet in the diameter which bisects it.

Let Pp be any chord, Qq a parallel chord adjacent to it, then Vv through their middle points will pass through the centre of the curve.

Join PQ and let it meet the diameter in T.

Then

$$VT : vT :: PV : Qv :: pV : qv,$$



$\therefore$ the secant pq also passes through T. Thus at whatever distance Qq may be from Pp, PQ, pq always intersect on the diameter CV. Hence when Qq approaches Pp and finally coincides with it so that PQ, pq, become finally the tangents at P, p, these tangents will meet in the diameter CV.

8. $CV . CT = CP^2$.

If QV be an ordinate of any diameter CP, and QT the tangent at Q, then CP is a mean proportional between CT and CV.

Let PR be the tangent at P intersecting QT in R, PR is parallel to QV: draw PO parallel to QT to meet QV in O.

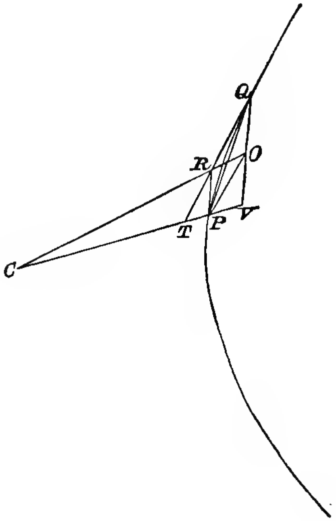
Join OR; it will be the diagonal of the parallelogram $OPRQ$ and will therefore bisect the chord QP; hence (by the last proposition) RO bisecting the chord PQ and passing through the intersection of the tangents at its extremities is a diameter, and when produced will pass through C.

Hence $\quad CV : CP :: CO : CR :: CP : CT$

and $\quad CV . CT = CP^2$. Q. E. D.

Cor. 1. If PN be the ordinate from P to the transverse axis, and the tangent at P intersects the axis in T, we shall have $\quad CN . CT = CA^2$.

Cor. 2. Hence to draw tangents to an hyperbola from any point T, join CT and produce it to meet the curve in P; draw the tangent at P by the aid of Cor. 1.

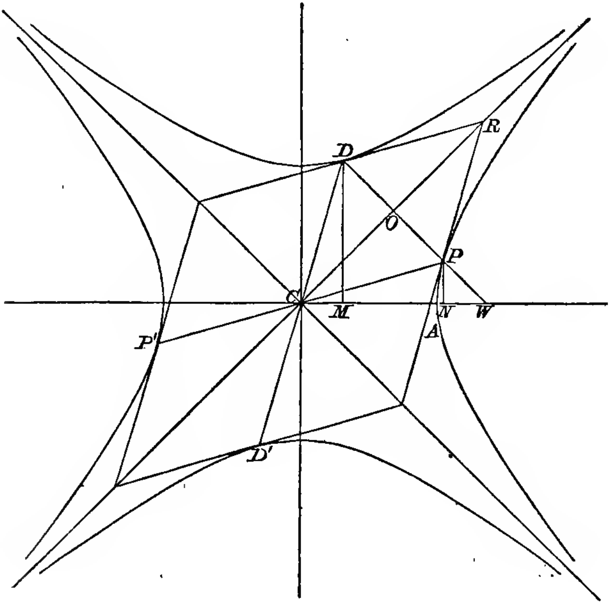


Produce CP to a point V such that CV is a third proportional to CT, CP: through V draw an ordinate QVq parallel to the tangent at P: join TQ, Tq; these (by the Proposition) will be the tangents required.

9. The Conjugate Hyperbola, Conjugate Diameters.

If on the conjugate axis of a rectangular hyperbola as transverse axis we draw another rectangular hyperbola, it is called the *conjugate hyperbola*, and it is manifest that it has the same asymptotes as the first.

Moreover if we draw semi-diameters *CP*, *CD* to meet the two curves equally inclined to the asymptotes and therefore to the axes, these will also be equal: and if we join *PD* it will cut the asymptotes at right angles, in *O* suppose. Take *OR* on the asymptote equal to *CO* and join *PR*, *DR*. Then *PR* = *CP* and therefore touches the hyperbola at *P*; and *DR* (= *CD*) touches the conjugate at *D*. Also *CPRD* is a parallelogram.



Moreover *CP* produced bisects all chords of the hyperbola parallel to *PR* or to *CD*, and similarly *CD* produced bisects all chords of the conjugate parallel to *DR* or to *CP*. Hence *CP*, *CD* produced each bisects the chords parallel to the other, and are called *conjugate* semi-diameters.

It will also be seen that CD will bisect the chords between the two branches of the curve parallel to PCP', and CP will bisect all chords of the conjugate parallel to DCD'.

10. The area of the parallelogram $CPRD$ is constant $= AC^2$.

Draw PN, DM at right angles to the transverse axis: the triangles CPN, CDM are equal in all respects. For the angles DCM, PCN are equally in excess and defect of half a right angle, and therefore together make up a right angle, as do also CPN and PCN; hence $DCM = CPN$ and $PCN = CDM$ and the triangles are equiangular and equal in all respects; $DM = CN$ and $CM = PN$.

But $PN^2 + AC^2 = CN^2$, hence also

$$CM^2 + AC^2 = CN^2 \text{ and } PN^2 + AC^2 = DM^2.$$

Now the triangle $DCP =$ quadrilateral $DMNP$ together with triangle DCM less the equal triangle $CPN =$ quadrilateral $DMNP$. Now produce DP to meet the axis in W, DWM is half a right angle, hence triangles DMW, PNW are half the squares on DM, PN, and $\frac{1}{2} PN^2 +$ quadrilateral $DMNP = \frac{1}{2} DM^2$.

But $PN^2 + AC^2 = DM^2$ or $\frac{1}{2} PN^2 + \frac{1}{2} AC^2 = \frac{1}{2} DM^2$,

$$\therefore \text{ quadrilateral } DMNP = \frac{1}{2} AC^2.$$

Hence parallelogram $DCPR =$ twice triangle $DCP =$ twice quadrilateral $DMNP = AC^2$.

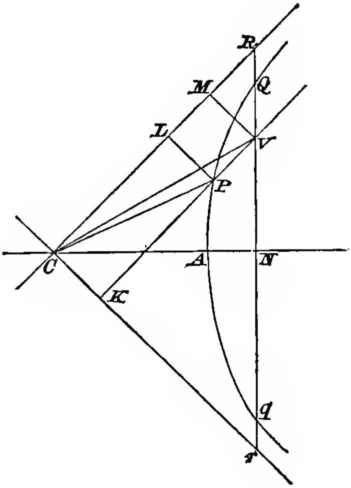
Cor. 1. The parallelogram formed by tangents at the extremities of P, P' and D, D' of the diameters PCP', DCD' will have the constant area $4AC^2$.

Cor. 2. The rectangle $PO \cdot OC =$ half the parallelogram $CPRD = \frac{1}{2} AC^2$.

Cor. 3. If PF be the perpendicular from P on the conjugate semi-diameter CD, we shall have $PF \cdot CD = CA^2$.

11. $QV \cdot Vq$ is proportional to PV.

In the rectangular hyperbola, if a double ordinate Qq to the transverse axis cuts PV, a line from any point of the curve parallel to the asymptote CR in V, then we shall



have $QV \cdot Vq$ proportional to PV. Join CP, CV, and we shall have $QV \cdot Vq = 4$ times the triangle CPV. For Qq is bisected in N,

$$\therefore \quad QV \cdot Vq + VN^2 = QN^2,$$

$$\therefore \quad QV \cdot Vq + VN^2 + AC^2 = QN^2 + AC^2 = RN^2.$$

but $RV.Vr+VN^2=RN^2$,

$$\therefore\ QV.Vq+AC^2=RV.Vr.$$

Let VP produced cut Cr in K; draw VM, PL at right angles to CR: then $RV:Vr::MV:Kr::MV:VK$,

$$\therefore\ RV.Vr:RV^2::MV.VK:MV^2,$$

or $RV.Vr:MV.VK::RV^2:MV^2$:

but $RV^2=2MV^2$, $MV.VK=$ twice the triangle CKV;

$\therefore\ RV.Vr=4$ times the triangle CKV,

and $AC^2=$ twice the rectangle $PL.LC=4$ times the triangle CKP,

$\therefore\ QV.Vq+4$ times the triangle $CKP=4$ times the triangle CKV,

$\therefore\ QV.Vq=4$ times the triangle CPV:

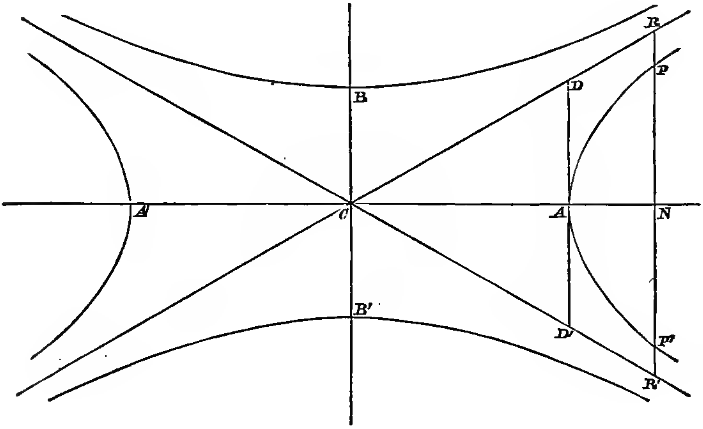
and since PV is parallel to CR, the triangle CPV is proportional to PV: hence also $QV.Vq$ is proportional to PV.

12. Now let us consider what modifications these propositions undergo when transferred by projection to the hyperbola of unequal axes.

Let us suppose the conjugate axis of any hyperbola less than the transverse.

Then we have shewn that the rectangular hyperbola whose axes are equal to the transverse axis will project into it if the planes of the two hyperbolas intersect in AA' and are inclined at the proper angle. Also the asymptotes will project into lines equally inclined to the transverse axis and at such an angle that if DAD' is drawn to them through A at right angles to CA, AD or AD' will $=BC$ (the conjugate semi-axis).

Also the conjugate rectangular hyperbola when projected will have BC for its transverse, CA for its conjugate semi-diameter. This is the conjugate hyperbola: both the hyperbola and its conjugate will continually approach the asymptotes CD, CD' without ever meeting them.



Also PR, $P'R'$, the distances between the curve and asymptotes on the double ordinate will be equal, and the tangents at A, A' will be at right angles to the transverse axis.

All lines perpendicular to the transverse axis will be diminished by projection in the ratio of $AC : BC$, hence $RP \cdot PR'$ which was equal to AC^2 will become BC^2.

Also V the middle point of any chord Uu of the asymptotes will also be the middle point of the chord of the curve as in § 5: and all chords to the curve parallel to Qq will be bisected by CV.

Also the tangent at P where CV cuts the curve will be parallel to QV and will be bisected at P: but as the angle WCw is now acute, each half of the tangent is less CP.

The proposition of § 6 must be modified by substituting the equal semi-conjugate diameter CD for CP; CD is parallel to Uu, and hence QV, CD, VU will be all altered in the same ratio by projection, and we shall have in the projected curve $QV^2 + CD^2 = QU^2$

or
$$QU \,.\, Qu = CD^2.$$

The proof of the properties of §§ 7, 8 applies equally to all hyperbolas, rectangular or not.

Referring to § 9 we see that CP, CD conjugate semi-diameters of the rectangular hyperbola will continue to bisect each other's chords after projection: and they are still said to be conjugate to one another. They will no longer be equally inclined to the asymptote, and whilst CM, CN will be unaltered, DM, PN will be diminished in ratio of $AC : BC$. Hence we shall have generally

$$CM^2 + CA^2 = CN^2 \text{ and } PN^2 + CB^2 = DM^2,$$

and by addition

$$PN^2 + CN^2 + CB^2 = DM^2 + CM^2 + CA^2,$$

or
$$CP^2 + CB^2 = CD^2 + CA^2.$$

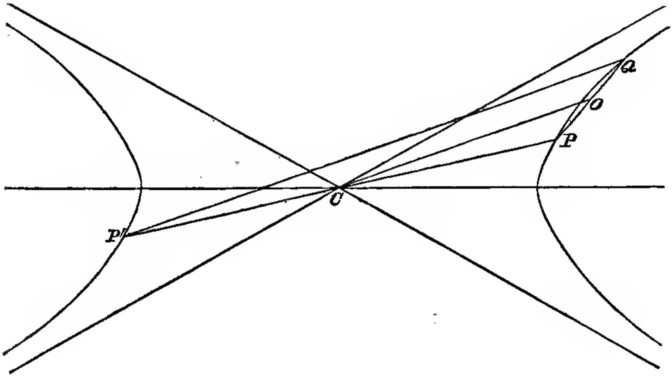
Also the parallelogram $CPRD$ will be a parallelogram after projection, and its area will be diminished in the ratio of $AC : BC$, and $\therefore = AC \,.\, BC$: and the parallelogram formed by tangents at the extremities of conjugate diameters will $= 4AC \,.\, BC$.

The proposition of § 11 will not be affected by projection, and therefore applies equally to the hyperbola of unequal axes, and it will be extended by a subsequent proposition

(Art. 15.) to the intercepts on a line drawn from any point in the curve parallel to an asymptote made by parallel lines in any direction.

13. If the conjugate axis of an hyperbola is greater than the transverse, we may project it into a rectangular hyperbola having its axes equal to the transverse, and the same results will obtain as in the last Article, except that the angles between the asymptotes that contain the transverse axis will be obtuse instead of acute.

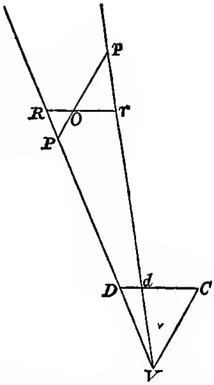
14. Supplemental Chords.



If any point Q in an hyperbola be joined with the extremities PP' of any diameter, PQ, $P'Q$ are called *supplemental chords*, and are parallel to conjugate diameters.

For if we take O the middle point of PQ and join CO, PQ will be parallel to the conjugate diameter to CO, but C is the middle point of PP' and therefore CO is parallel to $P'Q$, hence PQ, $P'Q$ are parallel to conjugate diameters.

15. The ratio of the rectangles Under the segments of two intersecting chords of an hyperbola is the same when one or each is moved into any position parallel to its former position.



Let VP, Vp be the generating lines of the cone through the extremities of one of the chords POp, O being the point of intersection of the two chords. Through V the vertex draw VC parallel to POp of some fixed length, the same for all the chords. Let planes through O and C at right angles to the axis of the cone intersect the plane $PVCp$ (the plane of the paper) in the lines ROr, DdC; and the generating lines VP, Vp in R, r, D, d; then Rr, Dd are chords of the circles in which these planes cut the cone.

Also Ror is parallel to DdC as well as POp to VC.

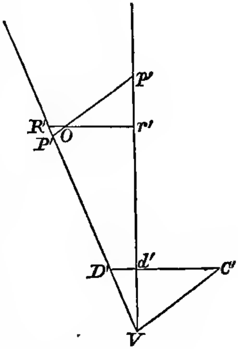
Hence by similar triangles POR, VCD,

$$PO : RO :: VC : CD,$$

and by similar triangles pOr, VCd,

$$Op : Or :: VC : Cd;$$

therefore $$PO \,.\, Op : RO \,.\, Or :: VC^2 : CD \,.\, Cd.$$



Now let $P'Op'$ be another chord of the hyperbola with its extremities on different generating lines of the cone, intersecting the former chord in O. Draw VC' parallel to $P'Op'$ and of the same length as VC: also $R'Or'$, $D'd'C'$ the intersections of planes through O and C' at right angles to the axis with the plane $P'VC'p'$. Then $R'Or'$ is a chord of the same circle as ROr was, and $RO.Or = R'O.Or'$, also $VC = VC'$: and as before $P'O \,.\, Op' : R'O \,.\, Or' :: VC'^2 : C'D' \,.\, C'd'$.

Hence we shall have

$$PO \cdot Op : P'O \cdot Op' :: C'D' \cdot C'd' : CD \cdot Cd.$$

Now observe that if the chords *Pop*, *P'op'* are moved parallel to themselves into some new position, but so as to be still chords of the hyperbola, the lines *VC*, *VC'* will not be affected, but the generating lines on which the extremities of the chords rest will not be the same as before; yet *C*, *C'* remaining fixed, *Dd*, *D'd'* in their new position will be still chords of the same circles as before, and each of the rectangles $CD \cdot Cd$, $C'D' \cdot C'd'$ will retain the same value as before. Hence the ratio $PO \cdot Op : P'O \cdot Op'$ will be invariable, and have the same value wherever *O* may be situated, provided the chords are drawn always parallel to their original position.

The same proof will hold when the extremities of the chords are on different branches of the curve, and when *O* is on the convex side of either branch.

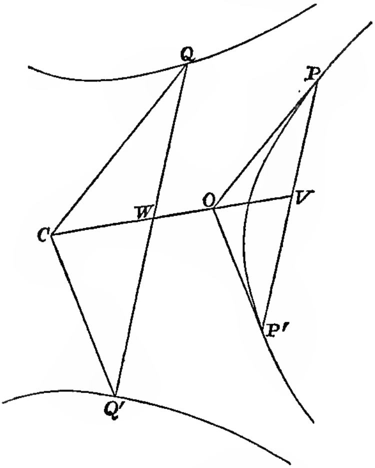
The same proof will also hold for the other sections of the cone equally with the hyperbola.

16. The ratio of the rectangles on the segments of the chords equals that of the squares on the parallel semi-diameters.

Let the chords move till they become tangents to the curve, viz. the tangents *OP*, *OP'* intersecting in *O*. The ratio of the rectangles is that of the squares on *OP*, *OP'*.

Draw *CQ* parallel to *OP* to meet the conjugate in *Q*, and draw *QQ'* parallel to *PP'* to meet *CQ'* parallel to *OP'* in *Q'*. Join *CO*, and let it meet *QQ'* in *W*, and when produced let it meet *PP'* in *V*.

Then V is the middle point of PP', and the triangles CWQ, CWQ' are similar to OVP, OVP':



$$\therefore\ QW : WC :: PV : VO,$$

and $$Q'W : WC :: P'V : VO,$$

$$\therefore\ QW : Q'W :: PV : P'V,$$

but $PV = P'V$; therefore also $QW = Q'W$.

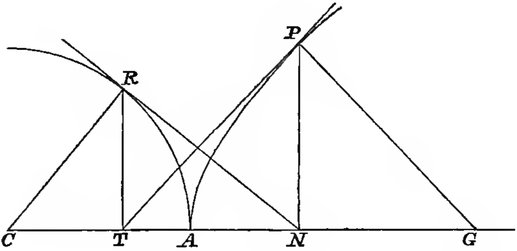
But CV bisecting PP', bisects all chords of the hyperbola and its conjugate parallel to PP', and therefore QQ', parallel to PP' and bisected by CO, is a chord of the conjugate hyperbola; CQ' is the semi-diameter parallel to OP', and the ratio of the rectangles under the segments of the chords $= OP^2 : OP'^2 = CQ^2 : CQ'^2$, since the triangles OPP', OQQ' are similar.

See also Besant's *Elementary Conic Sections*, p. 116.

Cor. If a circle intersect an hyperbola in four points, it may be easily shewn that the diameters parallel to the pairs of opposite sides of the quadrilateral formed by drawing the common chords will be equal, and therefore equally inclined to the axes of the curve; hence the opposite sides of the quadrilateral will be equally inclined to the axes.

17. $NG : NC :: BC^2 : AC^2$.

Let PG be the normal at P; draw NR from the foot of the ordinate PN to touch the auxiliary circle at R; then if



PT be the tangent at P, RT will be at right angles to the transverse axis, because $CN \,.\, CT = CA^2$. Also

$$NR^2 = NA \,.\, NA' \text{ and } \therefore NP^2 : NR^2 :: BC^2 : AC^2,$$

and

$$NP : NR :: BC : AC.$$

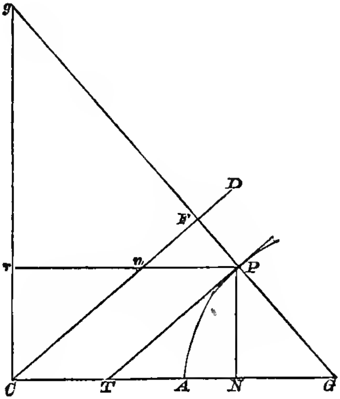
Join CR: then in the right-angled triangles TPG, CRN, $PN^2 = TN \,.\, NG$, $RN^2 = CN \,.\, TN$, $\therefore PN^2 : RN^2 :: NG : NC$;

$$\therefore NG : NC :: NP^2 : NR^2 :: BC^2 : AC^2.$$

18. $PF \,.\, PG = BC^2$, $PF \,.\, Pg = AC^2$.

If the normal at P intersects the axes in G and g, and PF be the perpendicular on CD the semi-diameter conjugate to CP, then first, $PF \,.\, Pg = AC^2$.

Let PT be the tangent at P, and let Pnr parallel to transverse axis cut CD and the conjugate axis in n and r, nT, rN will be parallelograms. Then since the angles at F and r are right angles, a circle will pass through F, n, r and g.



Hence $$PF \,.\, Pg = Pn \,.\, Pr = CT \,.\, CN = CA^2.$$

Also $$PG : Pg :: NG : NC :: BC^2 : AC^2:$$

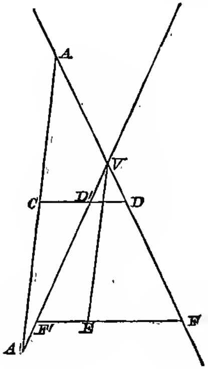
and $$PF \,.\, Pg = AC^2; \quad \therefore PF \,.\, PG = BC^2.$$

19. The asymptotes of any hyperbola are parallel to the generating lines of the cone in which a plane through the vertex parallel to the cutting plane cuts the cone.

Let VE be the line in which a plane through V parallel to the cutting plane cuts the plane of the paper.

Make VE equal to AC and draw FEF' through E at right angles to the axis.

Then the part of the perpendicular through E to the plane of the paper between that plane and the cone will be the height of a right-angled triangle which has half the angle between two generating lines at the base, call it EG. Then $EG^2 = EF \, . \, EF'$: and from equal triangles ACD, VEF and



$A'CD'$, VEF', $EF = CD$, $EF' = CD'$; $\therefore EG^2 = CD \, . \, CD' = BC^2$ and $EG = BC$. Hence the generating lines are inclined to VE at the same angle as the asymptotes to CA parallel to it, and being in a plane parallel to that containing the asymptotes, are parallel to them.

EXAMPLES.

1. Of all hyperbolas that can be cut from a given cone, the ratio $BC : AC$ is greatest in that cut by a plane parallel to the axis.

2. Give a construction for determining, if possible, the direction of the plane that will cut a rectangular hyperbola from a given cone.

3. NPQ is drawn from any point N of the conjugate axis of a rectangular hyperbola at right angles to it, to cut the auxiliary circle and the hyperbola in P and Q; prove $NP^2 + NQ^2 = 2AC^2$.

4. Prove the same for any hyperbola and the ellipse on the same axes.

5. A circle passes through A, A', the vertices of a rectangular hyperbola: the common chord parallel to AA' is a diameter of the circle.

6. The tangent PT to a circle intersects a fixed diameter in T, and TQ is drawn at right angles to the diameter of a length bearing a constant ratio to TP: as P moves on the circle, Q will move on an hyperbola which has the given circle for its auxiliary circle.

7. Through any point P of a circle on AB as diameter PA, BP are drawn and produced to Q and R such that QR is at right angles to BA and bisected by it: prove that as P moves on the curve Q, R will move on a rectangular hyperbola.

8. Draw a tangent to an hyperbola parallel to a given line.

9. From a given point in an hyperbola draw a line such that the intercept between the other point of intersection and an asymptote shall equal a given line. When does the problem become impossible?

10. The tangent at P meets an asymptote in T, and TQ is drawn to the curve parallel to the other asymptote. PQ produced both ways meets the asymptotes in R, R'. RR' is trisected in P, Q.

11. From a given point P in a rectangular hyperbola PM, PN are drawn equally inclined to an asymptote, and when produced meet the curve again in Q, R; prove that QR is a diameter.

12. Find the position and magnitude of the axes of an hyperbola which has a given line for asymptote, touches another line in a given point, and passes through another given point.

13. Draw lines from the centre of an hyperbola to the extremities of any chord: the intercepts of any line parallel to the chord between these lines and the asymptotes will be equal.

14. From $CV \cdot CT = CP^2$ deduce in the rectangular hyperbola, $VC \cdot VT = QV^2$.

15. In the rectangular hyperbola the triangles CVQ, QVT are similar.

16. In the rectangular hyperbola, R is the middle point of a chord Qq and RQ', Rq' are drawn parallel to the tangents at q, Q to meet CQ and Cq: shew that a circle will circumscribe $CQ'Rq'$.

17. From any point R of an asymptote RN, RM are drawn at right angles to the axes intersecting the hyperbola and its conjugate in P and D. Prove CP, CD are conjugate in the rectangular and general hyperbolas.

18. The tangent at P meets an asymptote in T, TN is drawn at right angles to the transverse axis; prove in the rectangular and general hyperbolas that NP passes through D the extremity of the diameter conjugate to CP.

19. Any two tangents have their points of intersection with the asymptotes joined: the lines so drawn will be parallel.

20. From any point P of an hyperbola PH, PK are drawn each parallel to one asymptote to meet the other: these lines produced if necessary meet any line through the centre in R and T. Complete the parallelogram $PRQT$, and shew that Q is a point on the curve.

21. Draw a tangent to an hyperbola at a given distance (less than AC) from C.

22. The tangent to an hyperbola meets a pair of conjugate diameters in T, t and the second tangents are drawn to the curve from T and t, they will touch it at the extremity of a diameter.

23. An hyperbola can be drawn through the extremities of any two radii of a circle having the diameters at right angles to the radii as asymptotes.

24. An hyperbola can be drawn through the extremities of any two semi-diameters of an ellipse having the diameters conjugate to them as asymptotes.

25. If PG be the normal at P, $CG = 2CN$ in the rectangular hyperbola.

26. $PG \, . \, Pg = CD^2$.

27. If the tangent at P intersects the asymptotes in R, r the circle on Gg as diameter will pass through C, R and r.

28. In the same hyperbola Gg varies inversely as the perpendicular from the centre on the tangent.

29. An hyperbola is cut from a given cone, and a straight line drawn from a point of it parallel to an asymptote; the plane through the vertex of the cone and this line will cut the cone in two straight lines one of which is parallel to the line in the curve.

Hence prove the proposition of Art. 11.

30. Apply the method of proof in Art. 15 to shew that if parallel tangents at Q, Q' meet the tangent at P in T, T',

$$QT : PT :: Q'T' : P'T'.$$

Prove the following propositions in the rectangular hyperbola:

31. The lines bisecting the angles between CP and the tangent at P are parallel to the asymptotes.

32. The tangent at the point Q intersects a pair of conjugate diameters in T, T': prove that CQ is the tangent to the circle round CTT'.

33. If V be the middle point of a chord, the lines bisecting the angles between CV and the chord are parallel to the asymptotes.

34. The lines bisecting the angles between supplemental chords are parallel to the asymptotes.

35. The angle between lines drawn from two points on the curve to one extremity of a diameter equals or is the supplement of that between the lines from the same points to the other extremity.

36. Diameters at right angles are equal to one another.

37. Of two chords of the curve at right angles to one another one has its extremities on the same branch, the other on different branches.

38. If AB, CD are chords at right angles to one another and the circle ABC cuts CD produced in E, BA will bisect ED.

39. The four circles that may be drawn through three of the points A, B, C, D in the last example consist of two pairs of equal circles.

40. If a tangent be at right angles to a chord, the circle on the chord as diameter will pass through the point of contact.

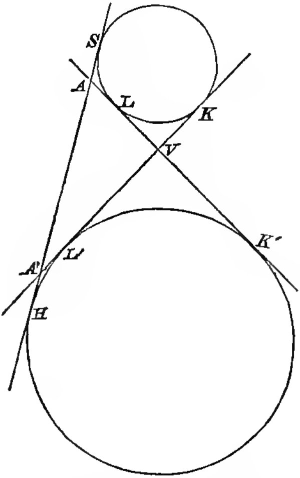
41. The line joining one end of a diameter with one end of a chord at right angles to it is at right angles to the line joining the other end of the diameter with the other end of the chord.

CHAPTER VII.

On the Focal properties of the Hyperbola.

1. Like the Ellipse, the Hyperbola has two foci and directrices: before treating of them let us premise the following propositions:

If the circles escribed on the sides VA, VA' of the triangle VAA' touch them and the produced sides in L, S, K, L', H, K' respectively, then $AS = A'H$ and KL' or $K'L = AA'$.



$$\begin{aligned}\text{For } SH &= SA + AH = AL + AK' \\ &= 2AL + LK',\end{aligned}$$

$$\text{also} = SA' + A'H = A'K + A'L' = 2A'L' + KL':$$

and $LK' = KL'$, $\quad\therefore AL = A'L'$ or $AS = A'H$.

Add AA' to each of these equals,

$$\therefore AS + AA' = AH = AK' = AL + LK' = AS + LK';$$

$$\therefore AA' = LK' \text{ or } KL'.$$

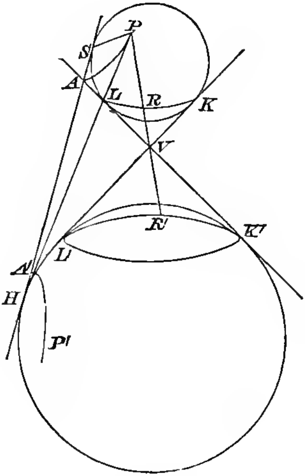
Also, if we produce KL, $K'L'$ to meet AA' in X, X' (as in the figure of the next article), $AX = A'X'$.

For since $AL = A'L'$ and these lines are equally inclined to the line joining the centre of the circles, their projections on this line will be equal: and AX, $A'X'$ have the same projections on the same line as AL, $A'L'$ and therefore have their projections on it equal, and being in the same straight line are themselves equal.

2. Now let AA' be the line in which a plane perpendicular to the plane of the paper cuts it. Let AP, $A'P'$ be the two branches of the hyperbola in which the same plane cuts the two sheets of the cone AVK, $A'VK'$ whose axis is the line joining the centres of the circles in the last Article: and let the circles SLK, $HL'K$ revolve about the axis, they will form spheres that touch the cone in circles LRK, $L'R'K'$.

Let P be any point in the hyperbola: the generating line PV of the cone will touch the spheres, in points R, R' suppose. Join SP, HP; these lines lie in the plane that touches the two spheres at S and H, and therefore touch them in those points: hence $SP = PR$, $HP = PR'$;

$\therefore HP - SP = PR' - PR = RR' = KL' = AA'$, a distance independent of the position of the point P: hence the difference of the distances of any point in the hyperbola from S and H is constant. S and H are called the *foci.*



3. Now let the plane through P perpendicular to the axis of the cone cut the cone in the circle QPQ' and the plane of the hyperbola in NP at right angles to $A'A$ produced. Then we shall have $SP = PR = QK$, and the triangles QAN, KAX are similar;

$$\therefore QK : NX :: AK : AX :: AS : AX.$$

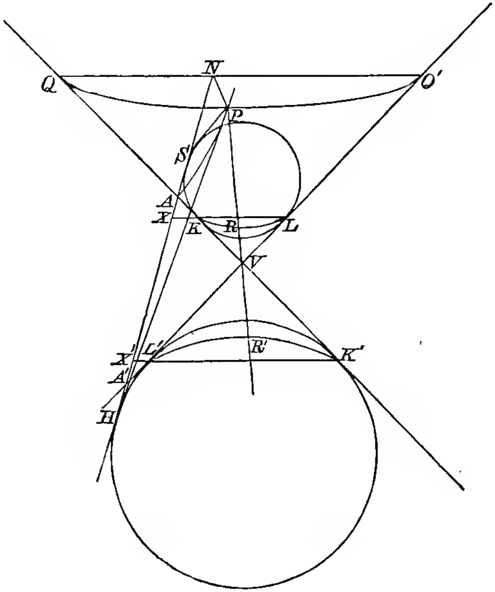
Hence, wherever P is situated on the curve $SP : NX$ is a constant ratio $= AS : AX$.

Similarly, joining HP,

$$HP = PR' = QK',$$

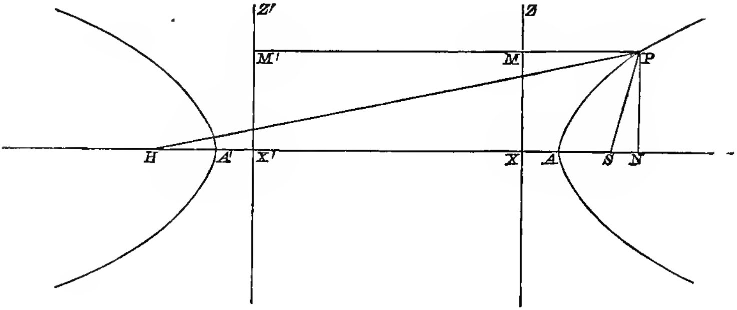
and the triangles QAN, $K'AX'$ are similar;

$$\therefore QK' : NX' :: AK' : AX' :: AK : AX :: AS : AX.$$



Hence also $HP : NX'$ is a constant ratio the same as $SP : NX$, wherever P is situated on the curve. We have shewn that $HA' : A'X' = SA : AX$: these ratios $=$ $A'L' : A'X'$ or $AK : AX$; and the triangles AXK, $A'X'L'$ are both right angled at X, or one of them has an obtuse angle at X; in either case we see that the ratios are ratios of greater inequality. As in the ellipse either of these ratios is called the *eccentricity*.

So then if we draw the hyperbolic section in the plane of the paper and draw through X, X' lines XZ, $X'Z'$ at right



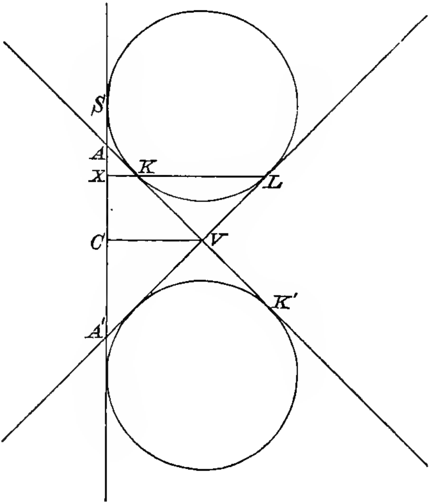
angles to XX' and PM, PM' perpendiculars to these lines, we have for any point P of the hyperbola

$$\left.\begin{array}{l} SP : NX \text{ (or } PM) \\ HP : NX' \text{ (or } PM') \end{array}\right\} :: AS : AX :: A'H : A'X'$$

ratios of greater inequality.

4. The position of the foci is determined relatively to the magnitude of the axes by the following relation $CS^2 = CA^2 + CB^2$.

We have shewn that by giving the proper angle to the cone any hyperbola of given axes may be cut from it by a plane parallel to the axis. Let then the given hyperbola be made by the intersection of a plane parallel to the axis of the cone. The foci will still be the points where the inscribed spheres touch the cutting plane, and these spheres will be equal. The figure represents the section of the cone and spheres by the plane of the paper.



CV will now be perpendicular to the axis and equal to CB. Also $AA' = KK' = 2KV$ since the circles are equal: $\therefore CA = KV$.

Hence $CS = CA + AS = KV + AK = AV$,

$$\therefore CS^2 = AV^2 = CA^2 + CV^2 = CA^2 + CB^2.$$

5. CX is a third proportional to CS and CA.

For in the last figure KX is parallel to CV,

$$\therefore AV : AC :: KV : XC,$$

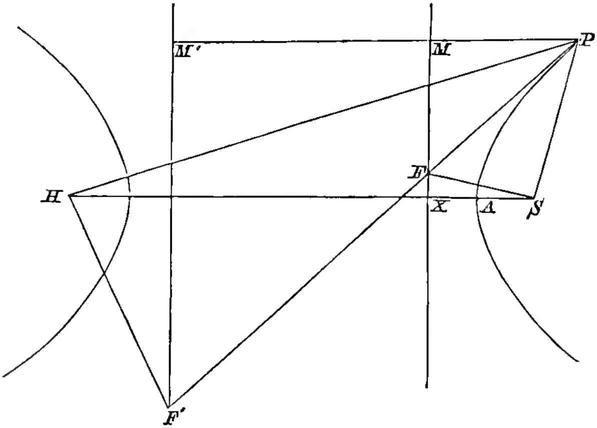
$$\text{or } CS : CA :: CA : CX.$$

Cor. Each of these ratios $= AK : AX$ or $AS : AX$, the eccentricity.

6. Properties of the secant and tangent.

Art. 6. of Chap. V. applies verbatim to all the Conic Sections.

7. The focal distances make equal angles with the tangent at any point.



Let the tangent at P meet the directrices in F, F', join SF, HF'; PSF, PHF' will be right angles.

Also the triangles MPF, $M'PF'$ are similar:

and $SP : PM :: AS : AX :: HP : PM'$,

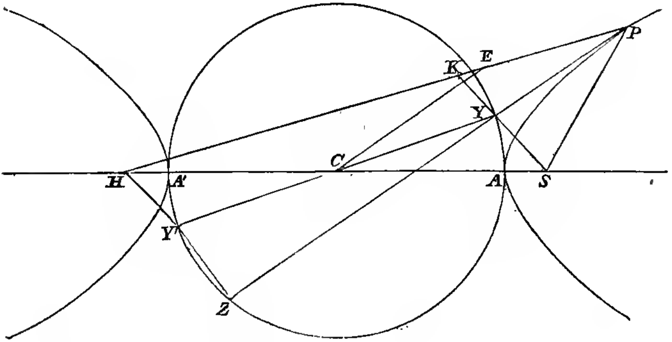
$\therefore SP : PH :: PM : PM' :: PF : PF'$,

or $SP : PF :: HP : PF'$.

Hence (Euclid VI. A.) SPF, HPF' are similar triangles, and the angles SPF, HPF' are equal.

Cor. The tangents at A, A' the vertices of the curve are at right angles to the transverse axes.

8. The feet of the perpendiculars from the foci on any tangent lie on the circle on AA' as diameter.



Let SY, the perpendicular from S on the tangent at P, meet HP in K: join SP, CY. Then in the right-angled triangles SYP, KYP, PY is common and the angle SPY = the angle HPY (by the last proposition).

Hence $SY = KY$, and $SP = KP$;

$\therefore KH = HP - KP = HP - SP = 2AC$.

And C, Y being the middle points of SH, SK, CY is parallel to HK and half of it and therefore $= AC$.

Hence Y lies on the circle on AA' as diameter. And similarly Z the foot of the perpendicular from H.

Cor. 1. CY being parallel to HP and bisecting SH, when produced will also bisect SP. Hence SYP being a right angle, the circle on SP as diameter passes through Y and has its centre on CY produced; hence it touches the auxiliary circle at Y.

Cor. 2. If CE be drawn parallel to the tangent at P

and therefore conjugate to CP and intersect PH in E, then $CEPY$ is a parallelogram and $PE = CY = AC$.

9. If SY, HZ are perpendiculars on the tangent at P from the foci S, H, $SY.HZ = BC^2$.

Since (in the figure of the last Article) YZH is a right angle, YC produced will meet ZH on the circumference of the circle.

The triangles SCY, HCY' are equal in all respects, and

$$SY.HZ = HY'.HZ = HA.HA' = HC^2 - A'C^2 = BC^2.$$

10. $$SP.HP = CD^2.$$

Art. 10 of Chap. V. applies verbatim to the hyperbola.

11. If from any point Q of the tangent PK perpendiculars QN, QT be drawn to the directrix and SP, then

$$ST : QN :: AS : AX.$$

12. Hence if QP, QP' be the tangents to an hyperbola from an exterior point Q, QP, QP' subtend equal angles at S.

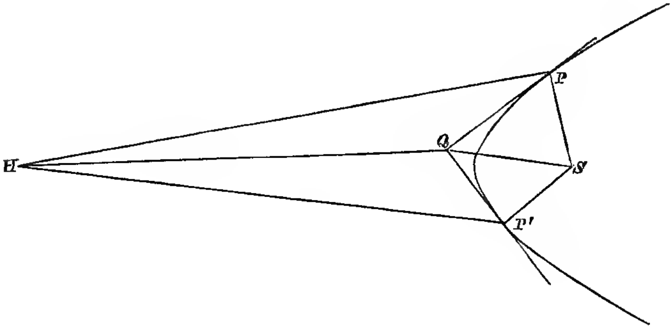
See Articles 11, 12 of Chap. V.

If P, P' are not on the same branch of the curve, since Q lies on the near side of the directrix to one branch and the further side to the other, T or T' will lie in PS or $P'S$ produced, and the supplement of one of the angles QSP, QSP' will be equal to the other, i.e. the angles will be supplementary.

Cor. Combining this last proposition with that of Art. 7 we see that the point Q is equidistant from the four lines SP, HP, SP', HP'.

13. If QP, QP' be the tangents to an hyperbola from a

point Q, the angles SQP, HQP' are supplementary, as are also SQP', HQP.



Join SP, SQ, SP', HP, HQ, HP'.

The angles of the quadrilateral $SPHP'$ are bisected by the lines drawn from the angular points to Q;

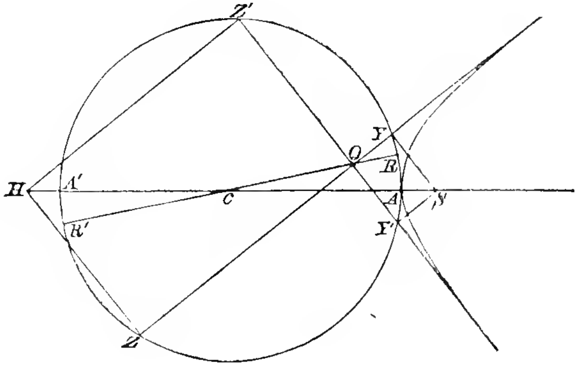
$\therefore$ the sum of the angles SPQ, PSQ, $HP'Q$, $P'HQ =$ sum of $P'SQ$, $SP'Q$, PHQ, HPQ.

But the first sum of angles with SQP, $HQP' =$ the sum of the interior angles of two triangles = the second sum with SQP', HQP: hence SQP, HQP' together $= SQP'$, HQP together: and these four angles together = four right angles: hence $SQP + HQP' = SQP' + HQP =$ two right angles. Q.E.D.

If P, P' lie on different branches it may be shewn, as in the case of the Ellipse, that SQP, HQP' are each half of the angle between HP and SP', and therefore (no longer supplementary but) equal to one another.

14. Any two tangents at right angles to one another intersect on a circle whose centre is C and square on the radius $+ BC^2 = AC^2$.

Let any two tangents at right angles to one another intersect in Q and cut the circle on AA' as diameter in Y, Z;



Y', Z'. Draw SY, SY'; HZ, HZ'. Let QC cut the circle in R, R'.

Then SQ, HQ will be rectangles with opposite sides equal.

Hence $$QR \cdot QR' = QY' \cdot QZ' = SY \cdot HZ = BC^2,$$

$$\text{and } CQ^2 + QR \cdot QR'^{2} = CR';$$

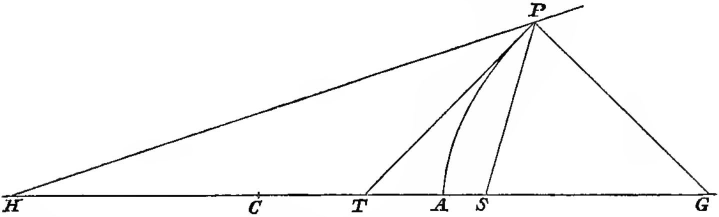
$$\therefore\ CQ^2 + BC^2 = AC^2:$$

and the locus of Q is a circle with centre C and square on its radius = difference of the squares on AC and BC.

Cor. If AC be less than BC the curve has no tangents at right angles to one another: but in this case the circle will be the locus of the intersections of tangents to the conjugate hyperbola at right angles to one another.

15. $SG : SP :: SA : AX.$

The tangent at P bisects the angle SPH: hence the



normal PG bisects the angle between SP and HP produced. Hence (Euclid VI. A)

$$SG : SP :: HG : HP :: HG - SG : HP - SP$$
$$:: 2SC : 2AC :: CS : CA :: SA : AX.$$

EXAMPLES.

1. Given a focus and two points of an ellipse, the other focus lies always on a certain hyperbola.

2. Four circles are drawn having the corners of a square for centres and diameters equal to the side; the locus of the centres of all circles which touch opposite pairs of circles, one internally and the other externally, is a pair of rectangular hyperbolas which have the common tangents of the circles for asymptotes.

3. The tangent from C to the circle through the foot of the directrix and the extremities of the latus rectum equals CS.

4. If N be the middle point of SX, prove $AN \cdot A'N = SN^2$.

5. The portion of either asymptote between the directrices equals the transverse axis.

6. Given the asymptotes and directrices, find the foci.

7. Given one asymptote and the direction of the second and a focus, find the vertices.

8. Given the asymptotes and one point on the curve, find the foci.

9. The focal distance of any point P on the hyperbola equals a line drawn from P parallel to an asymptote to meet the corresponding directrix.

10. Each of the tangents drawn to the auxiliary circle from the foci equals BC and touches it in a point where the directrix cuts it.

11. Two hyperbolas have the same asymptotes, shew that the chord of one touching the other is bisected at the point of contact.

12. Two hyperbolas have their foci coincident and the angles between the asymptotes supplementary: no tangent to one can be at right angles to a tangent to the other except the asymptotes.

13. The line drawn from S parallel to an asymptote to meet the curve equals a quarter of the latus rectum.

14. Two chords through the same focus have three of their four extremities on one branch, and the remaining extremity on the other: prove that the four lines joining their extremities intersect two and two on the corresponding directrix.

15. PQ a chord of one branch of an hyperbola subtends at S an angle double of that subtended at S by a length pq on the corresponding directrix: shew that Pp, Qq intersect on the curve.

16. An ellipse cuts any confocal hyperbola at right angles.

17. A circle is drawn through the foci and any point P of an hyperbola, the tangent and normal at P meet the conjugate axis in the same points as this circle.

18. P, Q are points in two confocal hyperbolas at which SH subtends equal angles, the tangents at P, Q are inclined at an angle equal to that subtended by PQ at either focus.

19. HY' drawn parallel to SP meets SY produced on the circumference of a fixed circle.

20. Given one focus, one point and one tangent of an hyperbola, the locus of the other focus is an hyperbola.

21. Qq a chord of the asymptotes moves parallel to itself and tangents are drawn from Q, q, their intersection will lie on a straight line through C.

22. A rectangular hyperbola confocal with an ellipse cuts it at the extremities of equal conjugate diameters of the ellipse.

23. An ellipse and confocal hyperbola intersect in P: one asymptote passes through the point of the auxiliary circle of the ellipse corresponding to P.

24. An ellipse and hyperbola have each their foci at the vertices of the other: if the tangents at the point of intersection meet the conjugate axis in t, t', $Ct = Ct'$.

25. The tangent at P is perpendicular to one asymptote and PS through the focus meets that asymptote in Q, prove $SQ = AC$.

26. P is the point of the hyperbola where SP is at right angles to HP: and they intersect CD in E, E'.

Prove $EE'^2 = 2AC^2$ and $CD^2 = 2BC^2$.

27. From the point of intersection of an asymptote and directrix a tangent is drawn to the hyperbola: the line joining the interior focus of the branch it touches with the point of contact is parallel to the asymptote.

28. The tangent from *P* a point in the asymptote touches the curve in *O* : *HT* is parallel to the same asymptote; prove that *HP* bisects the angle *THO* : and if *PO* intersects the other asymptote in *Q*, *PHQ* = half the angle *PCQ*.

29. The focal distances of two points *P*, *P′* intersect in *O*, prove that the tangents *QP*, *QP′* subtend equal angles at *O*.

30. If *SP*, *HQ* are parallel, find the locus of the intersection of the tangents at *P* and *Q*.

31. *PT*, *QT* tangents from *T* a point in the auxiliary circle whether to the same branch or not have one of the focal distances *SP*, *SQ* parallel to one of the two *HQ*, *HP*.

32. The tangent at *P* is perpendicular to two parallel tangents at *Q*, *Q′*, prove that *SQ*, *HQ′* subtend equal angles at *P*.

33. If *GL* be the perpendicular from *G* on *SP*, prove *GL* : *PN* a constant ratio.

34. From the vertex *A* draw *AQ* perpendicular to the tangent at *P*, and let *QA* produced meet *PS* produced in *O*, the locus of *O* is a circle.

35. Let *GY* produced meet *ZH* produced in *Z′*, then *HZ′* = *HZ*.

36. If the normals at *P*, *Q* extremities of a focal chord intersect in *O*, and *OL* parallel to the transverse axis cut *SP* in *L*, *L* is the middle point of *PQ*.

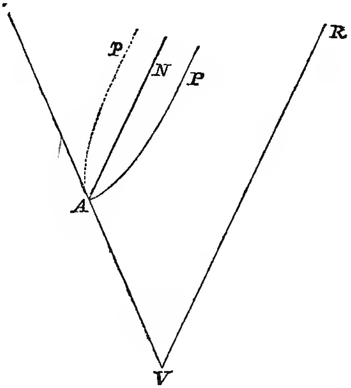
37. If *GR* be the perpendicular on *SP* from *G*, *PR* equals semi-latus rectum.

38. If the normal at *P* meets the axes in *G*, *g*, the triangles *SPg*, *GSg* are similar, and *Gg* : *Sg* and therefore *Pg* : *Sg* are constant ratios.

CHAPTER VIII.

On the Parabola.

1. We have now to consider the case in which the cutting plane is parallel to one of the generating lines of the cone. Let the cutting plane intersect the cone in the curve PAp, and intersect the plane of the paper which contains the axis in the line AN parallel to the generating line VR in the plane of the paper, then PAp is called a *Parabola.*



It is manifest that the parabola is divided into equal parts by the line *AN* which is called the axis, every line in the plane of the section at right angles to *AN* will meet the curve in two points on opposite sides of *AN* and at equal distances from it.

The axis intersects the curve in one point only, in other words the curve has only one vertex.

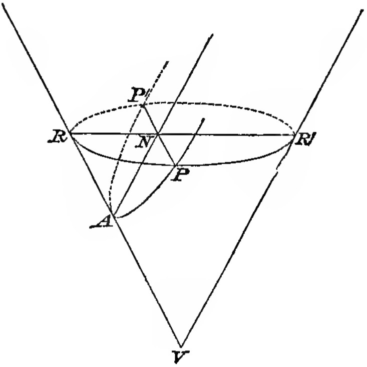
2. By turning the cutting plane about a line through *A* at right angles to the plane of the paper through the smallest possible angle, the parabola is changed either into an ellipse or hyperbola whose centre is at a great distance from *C*: the parabola may therefore be considered as the form to which the semi-ellipse cut off by the axis minor on the side of *A* and one branch of the hyperbola approach more and more nearly when, the vertex remaining fixed, the centre is removed to a greater and greater distance. Hence it appears that lines drawn from all points in the semi-ellipse and semi-hyperbola to their centres become more and more nearly parallel as the cutting plane moves towards the position in which it cuts a parabola from the cone.

And we may anticipate that the line joining the middle points of parallel chords of a parabola will be parallel to its axis; and generally all properties of the ellipse and hyperbola that relate to lines that remain finite in the parabola are equally true in that curve.

It will have been observed that there is a similarity in the properties of the ellipse and hyperbola; the parabola is the curve in which they meet, or in which each undergoes the transition into the other.

3. $NP^2 = 4AS . AN$.

Let RPR' be a circular section of the cone made by a plane at right angles to the axis through N, any point of



the axis of the parabola and cutting the plane of the curve in PNP' at right angles to the axis.

Then $NP^2 = RN . NR'$, and the ratio $NR : AN$ is constant for all positions of N.

Hence the ratio $NP^2 : AN . NR'$ is constant: and NR' is also invariable.

The ratio $RN : AN$ is invariable for all positions of N; the ratio $NR' : AN$ may be made to have all values from a ratio indefinitely great to one indefinitely small by moving N along the axis from A. Hence we may find a point S in the axis such that for that point $RN . NR' = 4AN^2$, or if LSL' be the double ordinate through S, $SL^2 = 4AS^2$ or $SL = 2AS$.

Hence generally

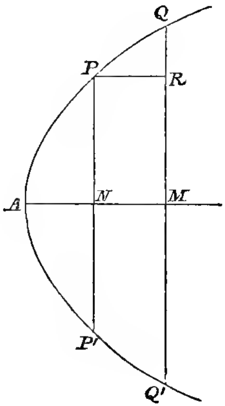
$$NP^2 : SL^2 :: AN . NR' : AS . NR' :: AN : AS,$$

$$NP^2 : 4AS^2 :: AN : AS,$$

$$\therefore NP^2 = 4AS . AN.$$

4. $QR . Q'R = 4AS . PR.$

Let PR from any point P of the curve meet the double ordinate QMQ' in R, then $QR . Q'R = 4AS . PR$. Draw the



double ordinate PNP', then PM is a rectangle whose opposite sides are equal: and since QQ' is bisected in M,

$$QR . Q'R + RM^2 = QM^2,$$

$$\text{or } QR . Q'R + PN^2 = QM^2;$$

$$\therefore QR . Q'R + 4AS . AN = 4ASAM;$$

$$\therefore QR . Q'R = 4AS . MN$$

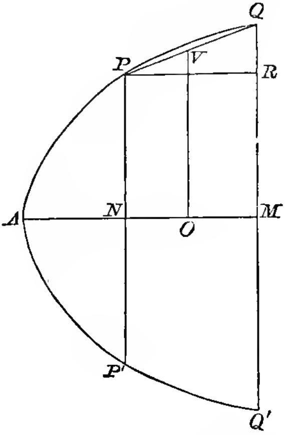
$$= 4AS . PR.$$

This proposition is equally true when, as in fig. § 6, R is a point in the chord QQ' external to the curve.

This property is seen in the case of the hyperbola in Chap. v. § 11. By Chap. vi. § 19 it appears that as the plane that cuts an hyperbola from the cone moves about the line through A towards the position in which it cuts a parabola, the angle between the asymptotes diminishes and they become more and more nearly parallel to the axis. Hence, as the hyperbola passes into the parabola, the line from any point of the curve parallel to an asymptote becomes parallel to the axis: and the proposition of the present Article takes the place of that previously proved for the hyperbola.

5. The middle points of parallel chords lie on a straight line parallel to the axis.

Let V be the middle point of any chord PQ, draw the ordinates PN, VO, and the double ordinate QMQ', PR parallel to the axis.



Then since QQ' is bisected in M,

$$RQ' = RM + MQ' = PN + QM = 2VO,$$

and $QR \cdot RQ' = 4AS \cdot PR$ or $QR \cdot VO = 2AS \cdot PR$,

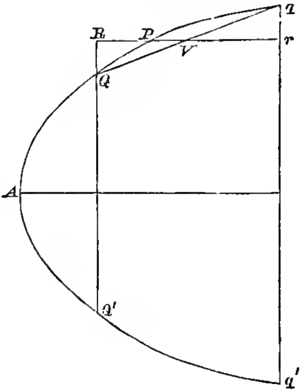
and $VO : 2AS :: PR : QR$ a constant ratio for all parallel chords. Hence VO is invariable for parallel chords, and all their middle points lie on a straight line parallel to the axis.

This line is called the *diameter* of the chords.

This property corresponds to the fact that the ellipse and hyperbola have the middle points of all parallel chords on a straight line through the centre.

6. $RQ^2 = 4AS \cdot PV$.

From V the middle point of any chord Qq draw VP parallel to the axis to meet the curve in P. Let PV produced both ways meet the double ordinates through Q, q (the former produced) in R, r: V will be the middle point of Rr,



and $Pr = PV + Vr = PV + VR = PR + 2PV$.

Also $qr = QR$, and $q'r = Q'R + 2QR$;

$$\therefore\ qr \,.\, q'r = QR \,.\, Q'R + 2QR^2,$$

$$\text{or}\ \ 4AS \,.\, Pr = 4AS \,.\, PR + 2QR^2,$$

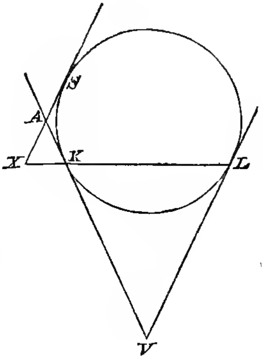
$$QR^2 + 2AS \,.\, PR = 2AS \,.\, Pr$$

$$= 2AS \,.\, PR + 4AS \,.\, PV;$$

$$\therefore\ QR^2 = 4AS \,.\, PV.$$

7. The point S we shall see is the focus of the curve, and we may now conveniently consider the properties of the curve with relation to it.

We premise the following proposition. If a circle touch the parallel lines VL, AS, and the line AV that intersects them in L, S and K, then if LK produced meets SA produced in X, $AS = AX$.



For $VK = VL$,

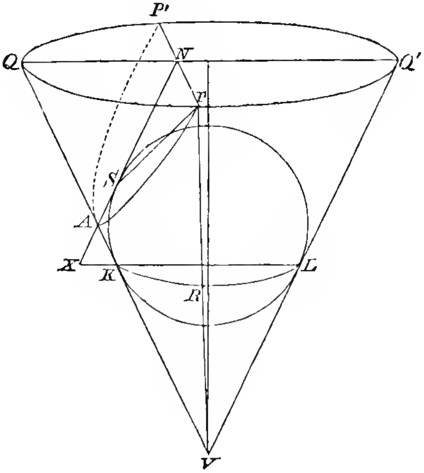
$\therefore$ the angle $VLK = VKL$: but VLK = alternate angle AXK, and VKL = vertical opposite angle AKX:

$\therefore$ $AXK = AKX$ and $AK = AX$, but $AS = AK$;

$\therefore$ $AS = AX$.

8. Now let AN be the intersection of the plane of the paper with a plane at right angles to it that cuts the cone

QVQ' in the parabola PAP', so that AN is parallel to VQ'.

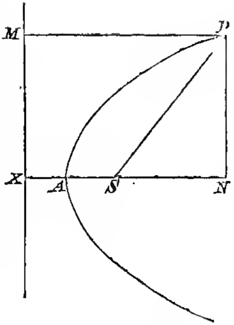


Describe a circle SKL in the plane of the paper touching the generating lines VQ, VQ' in K and L, and AN in S.

Then if we make the circle revolve about its diameter which coincides with the axis, it will generate a sphere which will touch the cone in the circle KRL. Every point of this circle will be equidistant from V. Let a circular section QPQ' through N cut the parabola in P and the plane of the paper in QNQ': the triangle QAN is similar to QVQ':

$$\therefore NA = QA: \text{ and } AX = AK, \quad \therefore NX = QK.$$

Join SP; also PV cutting the circle KRL in R. PR will $= QK$. And SP, PR tangents drawn to the sphere SKL from the point P are equal: $\therefore SP = PR = QK = NX$: hence the distance of any point of the curve from $S =$ the distance of the foot of its ordinate from X.



So then if we draw the parabolic section in the plane of the paper and draw through X a line XM at right angles to the axis, and PM at right angles to XM, then we have for any point of the curve $SP = NX = PM$: the distance of any point P from $S =$ its perpendicular distance from XM. As before, S is the *focus* and XM the *directrix* of the parabola.

The eccentricity of the parabola is thus seen to be a ratio of equality. This results also from the fact that the parabola is the limiting form of a semi-ellipse, or of one branch of the hyperbola, when the centre moves to a continually increasing distance from the vertex, whilst the focus approaches a position at a certain definite distance from the vertex.

In this case $AS : AX$ tends to become a ratio of equality. For in the ellipse and hyperbola $CS \cdot CX = CA^2$; therefore if CS, CX are the distances of an external point C from the points where a line from C through the centre of a circle cuts the circle, CA will be the length of the tangent: and as C moves to a continually increasing distance, the tangent will become more and more nearly parallel to the diameter through C, and CA will tend to become an arithmetic mean between CS, CX. Hence as the semi-ellipse or branch of the hyperbola passes into the parabola, the vertex will assume the position midway between S and X, and the eccentricity will become a ratio of equality.

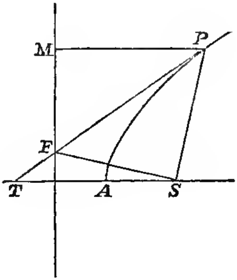
9. The position of S is identical with that previously assigned to it in proving the relation $PN^2 = 4AS \cdot AN$. For

SL the semi-latus rectum or ordinate through S will $= SX = 2SA$, which is the relation by which S was previously determined.

10. Properties of the secant and tangent. Art. 6 of Chap. IV. applies verbatim to the Parabola.

11. The tangent at any point makes equal angles with the focal distance of the point and the axis of the curve.

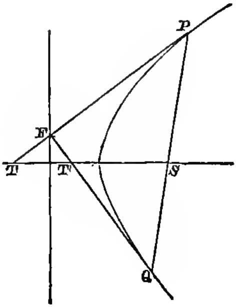
Let the tangent at P meet the directrix in F and the axis in T: join SP, SF, and draw the perpendicular PM. By the last Art. PSF is a right angle, also $SP = PM$; hence the right-angled triangles PSF, PMF having equal heights and hypotenuse common are equal in all respects: the angle $SPF = MPF = PTS$.



Cor. 1. Hence $SP = ST$, and the perpendicular on the tangent from S will bisect the angle PST.

Cor. 2. The angle between the focal distance and the tangent at any point cannot be a right angle except at the vertex, when SP, PM are in the same straight line and the tangent makes equal angles with them.

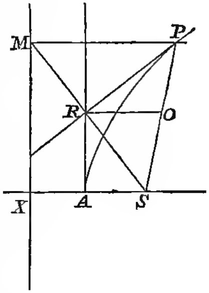
12. Tangents at the extremities of a focal chord intersect on the directrix in a right angle.



Let PF, QF, tangents at the extremities of the focal chord PSQ, intersect on the directrix in the point F and cut the axis in points T, T'.

Then the angle PFQ
$= FTT' + FT'T$
$= PTS + QT'S$
$= QPF + PQF$
= half the sum of the interior angles of the triangle FPQ
= a right angle.

13. The foot of the perpendicular from the focus on any tangent, lies on the tangent at the vertex.



Let PR the tangent at P intersect SM in R. Then in the triangles SPR, MPR, $SP = PM$, and the angle $SPR = MPR$, hence the triangles are equal in all respects and the angle $SRP = MRP$, SM is at right angles to PR. Join AR; then SX is bisected in A and SM in R:

$\therefore$ AR is parallel to XM and is perpendicular to the axis and touches the parabola at A the vertex. Hence the foot of the perpendicular on PR from S lies on the tangent at A.

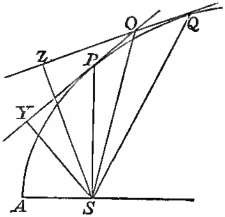
Cor. If O be the middle point of SP, RO will be parallel to MP, and therefore perpendicular to AR, and the circle on SP as diameter will touch AR at R. AR is the limiting form of the auxiliary circle in the ellipse and hyperbola when the centre removes to a continually increasing distance.

14. $ST : QN :: AS : AX$, as in Chap. IV. § 11.

15. The tangents to a parabola from an external point subtend equal angles at the focus, as in Chap. IV. § 12.

16. The angle between two tangents to a parabola equals that subtended by either tangent at the focus.

Let SY, SZ be the perpendiculars from the focus on the tangents to the parabola at P and Q which intersect in O. Join SP, SQ, SO. Then the angle $ASY =$ half ASP, and $ASZ =$ half ASQ.



Also the angle between the tangents equals the angle between the perpendiculars on them

$$= ZSY = ASZ - ASY$$
$$= \tfrac{1}{2}ASQ - \tfrac{1}{2}ASP = \tfrac{1}{2}PSQ$$
$$= OSP \text{ or } OSQ.$$

17. The triangles SOP, SQO are similar and $SO^2 = SP \,.\, SQ$.

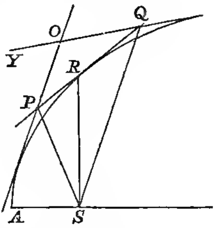
The same construction being made as in the last proposition, we shall have the sum of the angles SOP, $POZ = SOZ$ = the sum of the interior and opposite angles of the triangle SQO = the sum of SQO, OSQ.

But $POZ = OSQ$: $\therefore$ $SOP = SQO$.

And the angle $PSO = OSQ$. Hence the triangles SOP, SQO are similar and $SP : SO :: SO : SQ$ or $SO^2 = SP \cdot SQ$.

Cor. If SY be the perpendicular from S on the tangent at P, YA will be the tangent at A, and $SY^2 = SA \,.\, SP$.

18. The circle circumscribing the triangle formed by three tangents to a parabola passes through the focus.

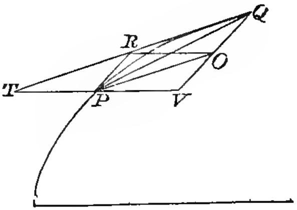


Let two of the tangents intersect in O and the third tangent cut them in P, Q and touch the curve in R. Join SR and produce QO to Y.

Then the angle $OPQ = PSR$,
and the angle $OQP = QSR$:
hence the angle $POY = OPQ + OQP$
$= PSR + QSR = QSP$:
the sum of the opposite angles POQ, PSQ of the quadrilateral $OPSQ$ = the sum of POQ, POY = two right angles, and the quadrilateral can be inscribed in a circle. In other words, the circle circumscribing the triangle POQ will pass through S.

19. The tangents at the extremities of a chord intersect on the diameter of the chord. See Chap. v. § 7.

20. $PT = PV$.



Let QT be the tangent at the point Q and QV the ordinate at that point to the diameter PV: to shew that $PT = PV$. For QV will be parallel to PR the tangent at P. Draw PO parallel to QT to meet QV in O. Join RO: being the diagonal of the parallelogram $POQR$ it will bisect the chord PQ which is the other diagonal.

Hence RO bisecting the chord PQ and passing through the intersection of the tangents at the extremities of the chord will be parallel to the axis of the curve and therefore to PV: $RPVO$, $RTPO$ are parallelograms and $PT = RO = PV$.

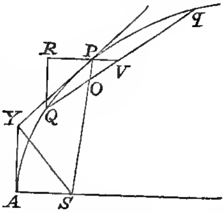
Cor. 1. If PN be the ordinate from any point P to the axis of the curve, and PT the tangent at that point, we shall have $AT = AN$.

Cor. 2. To draw the tangents to the parabola from an external point T, draw TPV parallel to the axis meeting the curve at P; make PV equal to PT, and through V draw the chord QVq parallel to the tangent at P: QT, qT will be the tangents required.

The proposition of this Article results from the corresponding $CV \,.\, CT = CP^2$ of the ellipse and hyperbola in the same way that $AS = AX$ results from the relation $CS \,.\, CX = CA^2$.

21. $QV^2 = 4\, SP \,.\, PV$.

We have already seen that if QR be the perpendicular distance of the extremity of a chord QVq from its diameter PV, $QR^2 = 4\, AS \,.\, PV$.



To find the relation between QR and QV draw SY the perpendicular on the tangent at P and join AY which will be the tangent at A. Join SP. Then the angle $SPY = RPY =$ the interior and opposite angle RVQ, since QV is parallel to PY; the right-angled triangles SPY, QVR are similar and $QV^2 : QR^2 :: SP^2 : SY^2$. But $SY^2 = AS \,.\, SP$;

$$\therefore \quad SP^2 : SY^2 :: SP^2 : AS \, . \, SP :: SP : AS;$$

$$\therefore \quad QV^2 : QR^2 :: SP : AS.$$

But $QR^2 = 4AS \, . \, PV, \quad \therefore \quad QV^2 = 4SP \, . \, PV.$

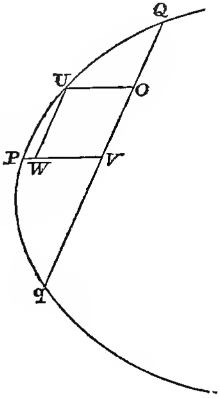
$4SP$ is called the *Parameter* of the diameter PV.

22. The length of the focal chord parallel to the tangent at $P = 4SP$.

Let SP intersect QV in O. Then QV being parallel to the tangent at P, makes equal angles with SP and PV; $\therefore$ the angle $POV = PVO$. $\therefore PV = PO$.

Hence if Qq passed through S we should have $PO = SP = PV$ and $QV^2 = 4SP^2$ or $QV = 2SP$, $Qq = 4SP$.

23. If two intersecting chords to a parabola move so as to be parallel to their former directions, the ratio of the rectangles under the segments is unaltered.



The method used in the case of the hyperbola (Chap. v. § 15) is equally applicable to the parabola. Or we may proceed thus:

Let Qq be one of two chords which intersect in O, PV its diameter. Draw OU, UW parallel to PV, Qq.

Then $QV^2 = 4SP \, . \, PV$, $UW^2 = 4SP \, . \, PW$.

$$\begin{aligned} \text{Or} \quad QV^2 &= 4SP \, . \, PV \\ &= 4SP \, . \, PW + 4SP \, . \, WV \\ &= UW^2 + 4SP \, . \, WV \\ &= VO^2 + 4SP \, . \, UO. \end{aligned}$$

But $QV^2 = QO \, . \, qO + VO^2, \quad \therefore \quad QO \, . \, qO = 4SP \, . \, UO.$

If $Q'Oq'$ be another chord through O, $P'V'$ its diameter we shall have $Q'O \,.\, q'O = 4SP' \,.\, UO$.

Hence $QO \,.\, qO : Q'O \,.\, q'O :: SP : SP'$.

When Qq, $Q'q'$ move parallel to themselves into any other position, P and P', the extremities of their diameters, remain fixed and SP, SP' are constant. Hence the ratio $QO \,.\, qO : Q'O \,.\, q'O$ is constant.

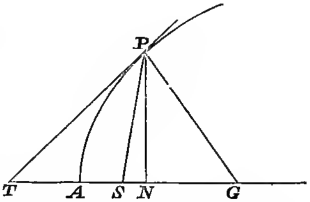
COROLLARY. From the relation $QO \,.\, qO = 4SP \,.\, UO$ we see that if a system of parallel chords cut a line drawn from any point in the curve parallel to the axis, the rectangle of their segments made by that line will equal the rectangle of the intercepts on it and a constant line.

This property is analogous to that of the hyperbola in Chap. IV. § 11.

For by Chap. IV. § 19 we see that when the axis of the hyperbola becomes more and more nearly parallel to a generating line of the cone, the asymptotes become more and more nearly parallel to that line and therefore to the axis. Hence the limiting direction of a line drawn parallel to the asymptote of an hyperbola when it passes into a parabola is the direction parallel to the axis.

24. $SG = SP$.

If PG be the normal to the parabola at the point P we shall have $SG = SP$.



For if PT be the tangent at P, $SP = ST$; and TPG is a right angle: hence TG is a diameter of the circle with centre S and distance SP or ST: $\therefore SG = SP$.

This corresponds to the proportion that holds, in the other sections of the cone $SP : SG :: AS : AX$: in the parabola $AS = AX$ and $SG = SP$.

25. $NG = 2AS$.

Let PN be the ordinate at P; PT, PG the tangent and normal at the same point. Then $NG = 2AS$.

For $TN = 2TA$, $TG = 2TS$.

$$\therefore\ TG - TN = 2TS - 2TA,$$

or $NG = 2AS$.

EXAMPLES.

1. A series of parabolas pass through two given points, and the axis is always parallel to the same line . prove that the focus will lie always on a certain hyperbola.

2. If two confocal parabolas intersect, their common chord passes through the intersection of the directrices, and bisects the angle between them.

3. Two parabolas have a common directrix, prove that their common chord bisects the line joining their foci at right angles.

4. PSQ is a focal chord of a parabola : PA, QA meet directrix in Y, Z. Prove that PZ, QY are parallel to the axis.

5. The normals at the extremities of a focal chord intersect in K. KL is perpendicular to the chord, KF parallel to the axis. Prove that F is the middle point of SL.

6. The tangent at P is parallel to the focal chord QSQ'; PV is its diameter; prove that the normal at P bisects VS.

7. Given two tangents and the directrix, find the focus.

8. Tangents at P, Q extremities of a focal chord intersect on the directrix in T. Normals at P, Q cut TS in U, V. Prove that $PQ^2 = TU \cdot TV$.

9. Given the focal chord PSQ and the focus S, find the vertex.

10. The normals at the extremities of the focal chord PSQ intersect in R. Shew that $PR^2 = SQ \cdot QP$.

11. OP, OQ are tangents drawn to the parabola from O; OR parallel to the axis to meet the curve in R. The part of the tangent at R between OP, OQ is bisected at R.

12. OP, OQ are tangents drawn to the parabola from O, OT perpendicular to OP, cuts the normal at Q in T; join OS, ST: OST is a right angle.

13. PN is the ordinate of the point P of the parabola: SY the perpendicular from the focus on the tangent at P: prove $YP = YN$.

14. Find a point such that the tangents from it and the focal distances of the points of contact may be a parallelogram.

15. Two parabolas have the same focus and axes coincident, the line SPQ from the focus cuts them in P and Q: prove that the tangents at P and Q are parallel.

16. A circle through S touches the parabola at P. MPK drawn from the directrix parallel to the axis meets the circle again in K: prove that MSK is a right angle.

17. Two parabolas have a common directrix, prove that the common tangents intersect in it.

18. A given straight line is a chord of a parabola and at one end a normal: the axis is given in direction: find the focus and directrix.

19. If the tangent at any point P intersect the latus rectum SL in R, prove $SL : SR :: PN : SP$.

20. PQ is a tangent bounded by tangents OR, OT: PV, QV are drawn parallel to OT, OR: shew that V lies on RT.

21. The locus of the middle points of focal chords is a parabola whose latus rectum = half that of the original curve.

22. PV is the diameter of a focal chord QSQ', QD meets PV produced at right angles in D: prove that $VD = VS$, and $QS \,.\, Q'S = QD^2$.

23. If the diameter OV of a focal chord meets the directrix in O prove that $SO^2 = 2AS \,.\, OV$.

24. A circle touches the directrix at M and the diameter from M cuts the curve in P: the diameter of the circle $= 4SP$. Shew that the common chord passes through S and that MP produced bisects it.

25. Two parabolas with a common vertex are turned in opposite directions on the same axis, and the focus of one is eight times as distant from the vertex as that of the other: the common tangent is drawn at the vertex. Every tangent to the first parabola has the part between the tangent at the vertex and the axis bisected by the other.

26. If a circle touch a parabola at P and cut it in two points Q, R: the tangent at P and the chord QR are equally inclined to the axis: and PO drawn to the point where the axis cuts the circle is parallel to QR, and therefore $PQ = OR$.

27. In a parabola two chords are equally inclined to the axis; if another parabola passes through the extremities of these chords, it will have its axis at right angles to that of the first parabola.

28. Two parabolas have a common focus and axis; their vertices are turned in opposite directions: a straight line from S cuts them in P and Q: prove that the tangents at P, Q are at right angles to one another.

29. Given two tangents and their points of contact, determine the focus and vertex.

30. OP, OQ are tangents to the parabola from O, prove

$$OP^2 : OQ^2 :: SP : SQ.$$

31. The lines joining the intersections of the tangents to confocal parabolas drawn to both curves at their points of intersection pass through the common focus.

32. OP, OQ are tangents to the parabola from O; if the chord PQ meets the directrix in F, prove that OSF is a right angle. If OK be at right angles to the directrix, prove SK is at right angles to PQ. If PQ cuts the axis in N, prove that KN is parallel to OS. If OM be drawn to meet the axis at right angles $AM = AN$.

33. OP, OQ are tangents to the parabola from O. Prove that when PQ moves parallel to itself, O moves parallel to the axis: when PQ moves round a point in the axis, O moves at right angles to the axis: when PQ moves round a point in the directrix, O moves in a straight line to S.

34. A line drawn from the focus to meet the tangent at a constant angle, has its point of intersection with it, on one of two fixed tangents.

35. Given one tangent to a parabola, to draw two others which make a given angle with it. In what case is one of the tangents removed to an infinite distance?

36. BC the portion of a tangent intercepted between two other tangents AB and AC is bisected by D the point of contact. Prove that SA is a fourth proportional to AD, AB and AC.

37. The portion of any tangent between tangents that meet on the directrix, subtends a right angle at the focus.

38. The tangent at P meets the directrix in F: from any point O in PF, and from F tangents are drawn to the curve, prove that they meet in the line through S at right angles to OS.

39. The chord PQ is a normal at P and QR is drawn parallel

to the axis to meet PP', the double ordinate through P, produced in R: prove that $PP' \,.\, P'R$ is constant.

40. The ordinate through the middle point of $NG = PG$.

41. An ellipse and parabola have a common focus and directrix: diagonals of the quadrilateral formed by joining the four points where the tangents at the extremities of the axis major cut the parabola pass through the focus and through the extremities of the axis minor.

42. An hyperbola is confocal with a parabola, and has the tangent at the vertex of the parabola for its nearer directrix. Prove that the tangent to the parabola at the point of intersection passes through the further vertex of the hyperbola.

MISCELLANEOUS EXAMPLES.

1. The orthogonal projection of a parabola is a parabola.

2. The projection of a parabolic section of a cone on a plane at right angles to the axis of the cone is a parabola having for its focus the point where the axis cuts the plane on which the projection is made.

3. $CS =$ the part of the generating line of the cone which has the same projection on the axis as CA has: this was proved for the ellipse, extend the proof to the hyperbola.

4. If an elliptic or hyperbolic section of a cone be projected on a plane through one vertex at right angles to the axis of the cone, CS^2 is diminished by the square on the distance of C from the plane of projection, and one focus of the projected curve will lie at the intersection of the axis of the cone with this plane.

5. All parabolas cut by parallel planes from a given cone have their foci on a straight line through the vertex of the cone.

6. Given a right cone and a point within it; only two sections have this point for focus and their planes are equally inclined to the line joining the point to the vertex.

7. Given the vertex of a cone and the centre of a sphere inscribed in it: all sections made by planes at right angles to a generating line and to the plane of the paper containing the centre and vertex, will have one of their foci on a circle which touches the axis of the cone at the centre of the sphere.

8. Two cones touch the same two spheres, prove that by whatever planes the two cones are cut, the ratio of their eccentricities is constant.

9. Two cones have supplementary angles and are placed with their vertices and one generating line of each coincident. Curves are cut from them by a plane at right angles to the coincident generating lines: shew that the directrices of either curve pass through the foci of the other.

10. The intersection of a plane with a cylinder is an ellipse with foci at the points of contact of the plane and two spheres inscribed in the cylinder.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

NOVEMBER, 1871.

A CATALOGUE of EDUCATIONAL BOOKS with a Short Account of their Character and Aim,

Published by

MACMILLAN AND CO

Bedford Street, Covent Garden, London.

CLASSICAL.

Æschylus.—ÆSCHYLI EUMENIDES. The Greek Text, with English Notes and English Verse, Translation, and an Introduction. By BERNARD DRAKE, M.A., late Fellow of King's College, Cambridge. 8vo. 3*s*. 6*d*.

The Greek text adopted in this Edition is based upon that of Wellauer, which may be said, in general terms, to represent that of the best manuscripts. But in correcting the Text, and in the Notes, advantage has been taken of the suggestions of Hermann, Paley, Linwood, and other commentators. In the Translation, the simple character of the Æschylean dialogues has generally enabled the author to render them without any material deviation from the construction and idioms of the original Greek

"The Notes are judicious, and, a rare merit in English Notes, not too numerous or too long. A most useful feature in the work is the Analysis of Müller's celebrated dissertations."—BRITISH QUARTERLY REVIEW.

Aristotle. — AN INTRODUCTION TO ARISTOTLE'S RHETORIC. With Analysis, Notes, and Appendices. By E. M. COPE, Senior Fellow and Tutor of Trinity College, Cambridge. 8vo. 14*s*.

This work is introductory to an edition of the Greek Text of Aristotle's Rhetoric, which is in course of preparation. Its object is to render that treatise thoroughly intelligible. The author has aimed to illustrate, as preparatory to the detailed explanation of the work, the general bearings and relations of the Art of Rhetoric in itself, as well as the special mode of treating it adopted by Aristotle in his peculiar system. The evidence upon obscure or doubtful questions connected with the subject is examined; and the relations which Rhetoric bears, in Aristotle's view, to the kindred art of Logic are fully considered. A connected Analysis of the work is given, sometimes in the form of paraphrase; and a few important matters are separately discussed in Appendices. There is added, as a general Appendix, by way of specimen of the antagonistic system of Isocrates and others, a complete analysis of the treatise called 'Ρητορικὴ πρὸς 'Αλέξανδρον, *with a discussion of its authorship and of the probable results of its teaching.*

ARISTOTLE ON FALLACIES; OR, THE SOPHISTICI ELENCHI. With a Translation and Notes by EDWARD POSTE, M.A., Fellow of Oriel College, Oxford. 8vo. 8*s*. 6*d*.

Besides the doctrine of Fallacies, Aristotle offers, either in this treatise or in other passages quoted in the commentary, various glances over the world of science and opinion, various suggestions or problems which are still agitated, and a vivid picture of the ancient system of dialectics, which it is hoped may be found both interesting and instructive. "It is not only scholarlike and careful, it is also perspicuous."—GUARDIAN. *"It is indeed a work of great skill."*—SATURDAY REVIEW.

Blackie.—GREEK AND ENGLISH DIALOGUES FOR USE IN SCHOOLS AND COLLEGES. By JOHN STUART BLACKIE, Professor of Greek in the University of Edinburgh. Fcap. 8vo. 2*s.* 6*d.*

"*Why should the old practice of conversing in Latin and Greek be altogether discarded?*"—PROFESSOR JOWETT.

Professor Blackie has been in the habit, as part of the regular training of his class in Edinburgh University, of accustoming the students to converse in Greek. This method he has found to be eminently successful as a means of furnishing the students with a copious vocabulary, training them to use it promptly, confidently, and with correct articulation, and instilling into them an accurate and intelligent knowledge of Greek Grammar. The method which has been so highly successful in Professor Blackie's hands, he believes, may be used with equal success by others; he has therefore in the present little volume furnished a series of twenty-five graduated Dialogues in parallel columns of Greek and English on a great variety of subjects, all of them calculated both to interest and instruct young men going through the usual course of School and College education in this Country. In the Preface, the Author fully explains the aim of the book, and the principle on which he himself intends to use it; where also, as well as in the Preliminary Remarks on Orthoepy, he gives a brief account of his theory of Greek Pronunciation, a theory which is now being gradually adopted by all the most eminent English scholars. The work has been revised by several eminent scholars, both English and Scotch. The GLOBE *says "Professor Blackie's system is sensible; his book is likely to be useful to teachers of Greek; and his suggestions valuable to the learners of any language."*

Cicero.—THE SECOND PHILIPPIC ORATION. With an Introduction and Notes, translated from the German of KARL HALM. Edited, with Corrections and Additions, by JOHN E. B. MAYOR, M.A., Fellow and Classical Lecturer of St. John's College, Cambridge. Third Edition, revised. Fcap. 8vo. 5*s.*

Cicero—*continued.*

This volume opens with a List of Books useful to the Student of Cicero, including History, Chronology, Lexicons, and some account of various editions, mostly German, of the works of Cicero. The Introduction is based on Halm: where Halm gives a reference to a classic, the passage has been commonly printed at length; where the reference is to Halm's notes on other Ciceronian speeches, or to modern books, the additional matter has been incorporated: and the numerous Greek quotations have been rendered into English. The English editor has further illustrated the work by additions drawn, for the most part, (1) *from the ancient authorities;* (2) *from his own private marginal references, and from collections;* (3) *from the notes of previous commentators. A copious 'argument' is also given. "On the whole we have rarely met with an edition of a classical author which so thoroughly fulfils the requirements of a good school-book."*—EDUCATIONAL TIMES. *"A valuable edition," says the* ATHENÆUM.

THE ORATIONS OF CICERO AGAINST CATILINA. With Notes and an Introduction. Translated from the German of Karl Halm, with many additions by A. S. WILKINS, M.A. Professor of Latin in Owens College, Manchester. Fcap. 8vo. 3*s.* 6*d.*

This edition is a reprint of the one prepared by Professor Halm for Orelli's Cicero. The historical introduction of Mr. Wilkins brings together all the details which are known respecting Catiline and his relations with the great orator. A list of passages where conjectures have been admitted into the text, and also of all variations from the text of Kayser (1862) *is added at the end. Finally the English Editor has subjoined a large number of notes, both original and selected from Curtius, Schleischer, Corssen, and other well-known critics, an analysis of the orations, and an index.*

Demosthenes.—DEMOSTHENES ON THE CROWN. The Greek Text with English Notes. By B. DRAKE, M.A., late Fellow of King's College, Cambridge. Fourth Edition, to which is prefixed ÆSCHINES AGAINST CTESIPHON, with English Notes. Fcap. 8vo. 5*s.*

An Introduction discusses the immediate causes of the two orations, and their general character. The Notes contain frequent references to the best authorities. Among the appendices at the end is a chronological table of the life and public career of Æschines and Demosthenes. "A neat and useful edition."—ATHENÆUM.

Hodgson.—MYTHOLOGY FOR LATIN VERSIFICATION. A brief Sketch of the Fables of the Ancients, prepared to be rendered into Latin Verse for Schools. By F. HODGSON, B.D., late Provost of Eton. New Edition, revised by F. C. HODGSON, M.A. 18mo. 3*s*.

The late Provost of Eton has here supplied a help to the composition of Latin Verse, combined with a brief introduction to Classical Mythology. In this new edition a few mistakes have been rectified; rules have been added to the Prosody; and a more uniform system has been adopted with regard to the help afforded.

Juvenal.—Thirteen Satires of JUVENAL. With a Commentary. By JOHN E. B. MAYOR, M.A., Fellow of St. John's College, Cambridge. Second Edition, enlarged. Part I. Crown 8vo. sewed. 3*s*. 6*d*.

The text is accompanied by a copious Commentary. For various notes the author is indebted to Professors Munro and Conington. All the citations have been taken anew from the original authors. "A painstaking and critical edition."—SPECTATOR. *"For really ripe scholarship, extensive acquaintance with Latin literature, and familiar knowledge of continental criticism, ancient and modern, it is unsurpassed among English editions."*—EDINBURGH REVIEW.

Marshall.—A TABLE OF IRREGULAR GREEK VERBS, classified according to the arrangement of Curtius' Greek Grammar. By J. M. MARSHALL, M.A., Fellow and late Lecturer of Brasenose College, Oxford; one of the Masters in Clifton College. 8vo. cloth. 1*s*.

The system of this table has been borrowed from the excellent Greek Grammar of Dr. Curtius.

Mayor (John E. B.)—FIRST GREEK READER. Edited after KARL HALM, with Corrections and large Additions by JOHN E. B. MAYOR, M.A. Fellow and Classical Lecturer of St. John's College, Cambridge. Second and Cheaper Edition. Fcap. 8vo. 4*s*. 6*d*.

A selection of short passages, serving to illustrate especially the Greek Accidence. A good deal of syntax is incidentally taught, and Madvig and other books are cited, for the use of masters: but no learner *is expected to know more of syntax than is contained in the Notes and Vocabulary. A preface "To the Reader," not only explains the aim and method of the volume, but also deals with classical instruction generally. The extracts are uniformly in the Attic dialect, and any Hellenistic forms occurring in the original classic authors, such as Ælian and Polybius, have been discarded in favour of the corresponding Attic expressions. This book may be used in connexion with Mayor's "Greek for Beginners." "After a careful examination we are inclined to consider this volume unrivalled in the hold which its pithy sentences are likely to take on the memory, and for the amount of true scholarship embodied in the annotations."*—EDUCATIONAL TIMES.

Mayor (Joseph B.)—GREEK FOR BEGINNERS. By the Rev. J. B. MAYOR, M.A., Professor of Classical Literature in King's College, London. Part I., with Vocabulary, 1*s*. 6*d*.; Parts II. and III., with Vocabulary and Index, 3*s*. 6*d*.; complete in one vol., fcap. 8vo. cloth, 4*s*. 6*d*.

The distinctive method of this book consists in building up a boy's knowledge of Greek upon the foundation of his knowledge of English and Latin, instead of trusting everything to the unassisted memory. The forms and constructions of Greek have been thoroughly compared with those of Latin, and no Greek words have been used in the earlier part of the book except such as have connexions either in English or Latin. Each step leads naturally on to its successor, grammatical forms and rules are at once applied in a series of graduated exercises, accompanied by ample vocabularies. Thus the book serves as Grammar, Exercise book, and Vocabulary. Where possible, the Grammar has been simplified; the ordinary ten declensions are reduced to three, which correspond to the *first three in Latin; and the system of* stems *is adopted. A general Vocabulary, and Index of Greek words, completes the work. "We know of no book of the same scope so complete in itself, or so well calculated to make the study of Greek interesting at the very commencement."*—STANDARD.

Peile (John, M.A.)—AN INTRODUCTION TO GREEK AND LATIN ETYMOLOGY. By JOHN PEILE, M.A., Fellow and Assistant Tutor of Christ's College, Cambridge, formerly Teacher of Sanskrit in the University of Cambridge. 8vo. 10*s.* 6*d.*

These Philological Lectures are the result of Notes made during the author's reading during the last three or four years. These Notes were put into the shape of lectures, delivered at Christ's College, during the last May term, as one set in the "Intercollegiate" list. They are now printed with some additions and modifications, but substantially as they were delivered. "The book may be accepted as a very valuable contribution to the science of language."—SATURDAY REVIEW.

Plato.—THE REPUBLIC OF PLATO. Translated into English, with an Analysis and Notes, by J. LL. DAVIES, M.A., and D. J. VAUGHAN, M.A. Third Edition, with Vignette Portraits of Plate and Socrates, engraved by JEENS from an Antique Gem. 18mo. 4*s.* 6*d.*

An introductory notice supplies some account of the life of Plato, and the translation is preceded by an elaborate analysis. "The translators have," in the judgment of the Saturday Review, *"produced a book which any reader, whether acquainted with the original or not, can peruse with pleasure as well as profit."*

Plautus (Ramsay).—THE MOSTELLARIA OF PLAUTUS. With Notes Critical and Explanatory, Prolegomena, and Excursus. By WILLIAM RAMSAY, M.A., formerly Professor of Humanity in the University of Glasgow. Edited by Professor GEORGE G. RAMSAY, M.A., of the University of Glasgow. 8vo. 14*s.*

"*The fruits of that exhaustive research and that ripe and well-digested scholarship which its author brought to bear upon everything that he undertook are visible throughout it. It is furnished with a complete apparatus of prolegomena, notes, and excursus; and for the use of veteran scholars it probably leaves nothing to be desired.*"—PALL MALL GAZETTE.

Potts (Alex. W., M.A.)—HINTS TOWARDS LATIN PROSE COMPOSITION. By ALEX. W. POTTS, M.A., late Fellow of St. John's College, Cambridge; Assistant Master in Rugby School; and Head Master of the Fettes College, Edinburgh. Second Edition, enlarged. Extra fcap. 8vo. cloth. 3*s.*

Those engaged in Classical teaching seem to be unanimously of the opinion that Composition in Latin Prose is not only the most efficient method of acquiring a mastery of the Latin language, but is in itself a valuable means of mental training, and an admirable corrective of some of the worst features in English writing. An attempt is here made to give students, after they have mastered ordinary syntactical rules, some idea of the characteristics of Latin Prose and the means to be employed to reproduce them. Some notion of the treatment of the subject may be gathered from the 'Contents.' CHAP. I.—*Characteristics of Classical Latin, Hints on turning English into Latin;* CHAP. II.—*Arrangement of Words in a Sentence;* CHAP. III.—*Unity in Latin Prose, Subject and Object;* CHAP. IV.—*On the Period in Latin Prose;* CHAP. V.—*On the position of the Relative and Relative Clauses.*

The GLOBE *characterises it as "an admirable little book which teachers of Latin will find of very great service."*

Roby.—A GRAMMAR OF THE LATIN LANGUAGE, from Plautus to Suetonius. By H. J. ROBY, M.A. late Fellow of St. John's College, Cambridge. Part I. containing:—Book I. Sounds. Book II. Inflexions. Book III. Word-Formation. Appendices. Crown 8vo. 8*s.* 6*d.*

This work is not a compilation from other Latin Grammars, but the result of an independent and careful study of the writers of the strictly *classical period, the period embraced between the time of Plautus and that of Suetonius. The author's aim has been to give the facts of the language in as few words as possible.* 1 *By* Grammar *the author means an orderly arrangement of the facts which concern the* form *of a language, as a Lexicon gives those which concern its* matter. 2. *This is a Grammar strictly of the* Latin *language; not a Universal Grammar illustrated from Latin, nor the Latin section of a Comparative Grammar of the Indo-European languages, nor a Grammar of the group of Italian dialects, of which Latin is one.* 3. *This is a Grammar of Latin from* Plautus *to* Suetonius, *with the latter of whom, the author believes, the silver age at latest ends. It will be found that the arrangement of the book and the treatment of the various divisions differ in many respects from those of previous grammars. Mr. Roby has given special prominence to the treatment of Sounds and Word-formation; and in the First Book he has done much towards settling a discussion which is at present largely engaging the attention of scholars, viz., the pronunciation of the classical languages. The author's reputation as a scholar and critic is already well known, and the publishers are encouraged to believe that his present work will take its place as perhaps the most original, exhaustive, and scientific Grammar of the Latin language that has ever issued from the British press.*

Sallust.—CAII SALLUSTII CRISPI CATILINA ET JUGURTHA. For Use in Schools. With copious Notes. By C. MERIVALE, B.D. (In the present Edition the Notes have been carefully revised, and a few remarks and explanations added.) Second Edition. Fcap. 8vo. 4*s.* 6*d.*

This edition of Sallust, prepared by the distinguished historian of Rome, contains an introduction, concerning the life and works of Sallust, lists of the Consuls, and elaborate notes. "A very good edition, to which the Editor has not only brought scholarship but independent judgment and historical criticism."—SPECTATOR.

The JUGURTHA and the CATILINA may be had separately, price 2*s.* 6*d.* each.

Tacitus.—THE HISTORY OF TACITUS TRANSLATED INTO ENGLISH. By A. J. Church, M.A., and W. J. Brodribb, M.A. With Notes and a Map. 8vo. 10*s.* 6*d.*

The translators have endeavoured to adhere as closely to the original as was thought consistent with a proper observance of English idiom. At the same time, it has been their aim to reproduce the precise expressions of the author. The campaign of Civilis is elucidated in a note of some length, which is illustrated by a map, containing the names of places and of tribes occurring in the work. There is also a complete account of the Roman army as it was constituted in the time of Tacitus. This work is characterised by the Spectator *as "a scholarly and faithful translation."*

THE AGRICOLA AND GERMANIA OF TACITUS. A Revised Text, English Notes, and Maps. By A. J. Church, M.A., and W. J. Brodribb, M.A. Fcap. 8vo. 3*s.* 6*d.*

"*We have endeavoured, with the aid of recent editions, thoroughly to elucidate the text, explaining the various difficulties, critical and grammatical, which occur to the student. We have consulted throughout, besides the older commentators, the editions of Ritter and Orelli, but we are under special obligations to the labours of the recent German editors, Wex and Kritz." Two Indexes are appended,* (1) *of Proper Names,* (2) *of Words and Phrases explained. "A model of careful editing," says the* Athenæum, "*being at once compact, complete, and correct, as well as neatly printed and elegant in style.*"

THE AGRICOLA and GERMANIA may be had separately, price 2*s.* each.

THE AGRICOLA AND GERMANIA. Translated into English by A. J. Church, M.A., and W. J. Brodribb, M.A. With Maps and Notes. Extra fcap. 8vo. 2*s.* 6*d.*

The translators have sought to produce such a version as may satisfy scholars who demand a faithful rendering of the original, and English readers who are offended by the baldness and frigidity which commonly disfigure translations. The treatises are accompanied by introductions.

notes, maps, and a chronological summary. The Athenæum *says of this work that it is "a version at once readable and exact, which may be perused with pleasure by all, and consulted with advantage by the classical student."*

Theophrastus.—THE CHARACTERS OF THEOPHRASTUS. An English Translation from a Revised Text. With Introduction and Notes. By R. C. JEBB, M.A., Public Orator in the University of Cambridge. Extra fcap. 8vo. 6*s.* 6*d.*

To the average English reader Theophrastus is little known. At the present time, when there is a general desire to see ancient life more vividly on every side from which it can illustrate our own, it seems possible that the characters of Theophrastus may possess some potent interest. The text has undergone careful revision. An Introduction supplies an account of the origin of the book, and of writers who have imitated it: as Hall, Sir Thomas Overbury, and others. The notes are for the most part selected from ancient sources. The SATURDAY REVIEW *speaks of it as "a very handy and scholarly edition of a work which till now has been beset with hindrances and difficulties, but which Mr. Jebb's critical skill and judgment have at length placed within the grasp and comprehension of ordinary readers."*

Thring.—Works by the Rev. E. THRING, M.A., Head Master of Uppingham School.

A LATIN GRADUAL. A First Latin Construing Book for Beginners. New Edition, enlarged, with Coloured Sentence Maps. Fcap. 8vo. 2*s.* 6*d.*

The Head Master of Uppingham has here sought to supply by easy steps a knowledge of grammar, combined with a good Vocabulary. Passages have been selected from the best Latin authors in prose and verse. These passages are gradually built up in their grammatical structure, and finally printed in full. A short practical manual of common mood constructions, with their English equivalents, forms a second part. To the New Edition a circle of grammatical constructions with a glossary has been added; as also some coloured Sentence Maps by means of which the different parts of a sentence can easily be distinguished, and the practice of dissecting phrases carried out with the greatest benefit to the student.

Thring—*continued.*

A MANUAL OF MOOD CONSTRUCTIONS. Fcap. 8vo. 1*s.* 6*d.*

Treats of the ordinary mood constructions, as found in the Latin, Greek, and English languages. THE EDUCATIONAL TIMES *thinks it "very well suited to young students."*

A CONSTRUING BOOK. Fcap. 8vo. 2*s.* 6*d.*

Thucydides.—THE SICILIAN EXPEDITION. Being Books VI. and VII. of Thucydides, with Notes. A New Edition, revised and enlarged, with a Map. By the Rev. PERCIVAL FROST, M.A., late Fellow of St. John's College, Cambridge. Fcap. 8vo. 5*s.*

This edition is mainly a grammatical one. Attention is called to the force of compound verbs, and the exact meaning of the various tenses employed. "The notes are excellent of their kind. Mr. Frost seldom passes over a difficulty, and what he says is always to the point."—EDUCATIONAL TIMES.

Virgil.—THE WORKS OF VIRGIL RENDERED INTO ENGLISH PROSE, with Introductions, Running Analysis, and an Index, by JAMES LONSDALE, M.A. and SAMUEL LEE, M.A. Globe 8vo. 3*s.* 6*d.*; gilt edges, 4*s.* 6*d.*

The preface of this new volume informs us that "the original has been faithfully rendered, and paraphrase altogether avoided. At the same time, the translators have endeavoured to adapt the book to the use of the English reader. Some amount of rhythm in the structure of the sentence has been generally maintained; and, when in the Latin the sound of the words is an echo to the sense (as so frequently happens in Virgil), an attempt has been made to produce the same result in English." The general introduction gives us whatever is known of the poet's life, an estimate of his genius, an account of the principal editions and translations of his works, and a brief view of the influence he has had on modern poets; special introductory essays are prefixed to the "Eclogues," "Georgics," and "Æneid." The text is divided into sections, each of which is headed by a concise analysis of the subject; the index contains references to all the characters and events of any importance.

Wright.—Works by J. WRIGHT, M.A., late Head Master of Sutton Coldfield School.

HELLENICA; OR, A HISTORY OF GREECE IN GREEK, as related by Diodorus and Thucydides; being a First Greek Reading Book, with explanatory Notes, Critical and Historical. Third Edition, with a Vocabulary. 12mo. 3*s*. 6*d*.

In the last twenty chapters of this volume, Thucydides sketches the rise and progress of the Athenian Empire in so clear a style and in such simple language, that the editor has doubts whether any easier or more instructive passages can be selected for the use of the pupil who is commencing Greek. This book includes a chronological table of the events recorded. THE GUARDIAN *speaks of the work as "a good plan well executed."*

A HELP TO LATIN GRAMMAR; or, The Form and Use of Words in Latin, with Progressive Exercises. Crown 8vo. 4*s*. 6*d*.

This book is not intended as a rival to any of the excellent Grammars now in use; but as a help to enable the beginner to understand them.

THE SEVEN KINGS OF ROME. An Easy Narrative, abridged from the First Book of Livy by the omission of Difficult Passages; being a First Latin Reading Book, with Grammatical Notes. With Vocabulary and Exercises. Fourth Edition. Fcap. 8vo. 5*s*.

This work is intended to supply the pupil with an easy construing book, which may at the same time be made the vehicle for instructing him in the rules of grammar and principles of composition. The notes profess to teach what is commonly taught in grammars. It is conceived that the pupil will learn the rules of construction of the language much more easily from separate examples, which are pointed out to him in the course of his reading, and which he may himself set down in his note-book after some scheme of his own, than from a heap of quotations amassed for him by others. "The Notes are abundant, explicit, and full of such grammatical and other information as boys require."—ATHENÆUM. "*This is really,*" *the* MORNING POST *says,* "*what its title imports, and we*

Wright—*continued.*

believe that its general introduction into Grammar Schools would not only facilitate the progress of the boys beginning to learn Latin, but also relieve the Masters from a very considerable amount of irksome labour . . . . a really valuable addition to our school libraries."

Or, separately,

SEVEN KINGS OF ROME. 3*s.*

VOCABULARY AND EXERCISES TO "THE SEVEN KINGS." 2*s.* 6*d.*

FIRST LATIN STEPS; OR, AN INTRODUCTION BY A SERIES OF EXAMPLES TO THE STUDY OF THE LATIN LANGUAGE. Crown 8vo. 5*s.*

*The aim of the author of this book is to put into the hands of pupils that which he thinks it needful for them to know before they commence a Latin author. The following points in the plan of the work may be noted:—*1. *The pupil has to deal with only one construction at a time.* 2. *This construction is made clear to him by an accumulation of instances.* 3. *As all the constructions are classified as they occur, the construction in each sentence can be easily referred to its class.* 4. *As the author thinks the pupil ought to be thoroughly familiarized, by a repetition of instances, with a construction in a foreign language, before he attempts himself to render it in that language, the present volume contains only Latin sentences.* 5. *The author has added to the Rules on Prosody in the last chapter, a few familiar lines from Ovid's Fasti by way of illustration; if these are translated, scanned, and learnt by heart, the pupil will be in a condition to derive from the practice of Latin versification all the good which it is calculated to afford. In a brief Introduction the author states in a clear, intelligible, interesting manner, the rationale of the principal points of Latin Grammar. Copious Notes are appended, to which reference is made in the text. From the clear and rational method adopted in the arrangement of this elementary work, from the simple way in which the various rules are conveyed, and from the abundance of examples given, both teachers and pupils will find it a valuable help to the learning of Latin.*

CLASSIC VERSIONS OF ENGLISH BOOKS, AND LATIN HYMNS.

THE following works are, as the heading indicates, classic renderings of English books. For scholars, and particularly for writers of Latin Verse, the series has a special value. The Hymni Ecclesiæ are here inserted, as partly falling under the same class.

Church (A. J., A.M.)—HORÆ TENNYSONIANÆ, sive Eclogae e Tennysono. Latine redditæ. Cura A. J. CHURCH, A.M. Extra fcap. 8vo. 6*s*.

Latin versions of Selections from Tennyson. Among the authors are the Editor, the late Professor Conington, Professor Seeley, Dr. Hessey, Mr. Kebbel, and other gentlemen.

Latham.—SERTUM SHAKSPERIANUM, Subnexis aliquot aliunde excerptis floribus. Latine reddidit Rev. H. LATHAM, M.A. Extra fcap. 8vo. 5*s*.

Besides versions of Shakspeare this volume contains, among other pieces, Gray's "Elegy," Campbell's "Hohenlinden," Wolfe's "Burial of Sir John Moore," and selections from Cowper and George Herbert.

Lyttelton.—THE COMUS OF MILTON, rendered into Greek Verse. By LORD LYTTELTON. Extra fcap. 8vo. 5*s*.

Lyttleton—*continued.*

THE SAMSON AGONISTES OF MILTON, rendered into Greek Verse. By LORD LYTTELTON. Extra fcap. 8vo. 6*s.* 6*d.*

"*Classical in spirit, full of force, and true to the original.*"—GUARDIAN.

Merivale.—KEATS' HYPERION, rendered into Latin Verse. By C. MERIVALE, B.D. Second Edit. Extra fcap. 8vo. 3*s.* 6*d.*

Newman.—HYMNI ECCLESIÆ. Edited by Rev. DR. NEWMAN. Extra fcap. 8vo. 7*s.* 6*d.*

Hymns of the Mediæval Church. The first Part contains selections from the Parisian Breviary; the second from those of Rome, Salisbury, and York.

Trench (Archbishop). — SACRED LATIN POETRY, chiefly Lyrical, selected and arranged for Use; with Notes and Introduction. Fcap. 8vo. 7*s.*

In this work the editor has selected hymns of a catholic religious sentiment that are common to Christendom, while rejecting those of a distinctively Romish character.

MATHEMATICS.

Airy.—Works by G. B. AIRY, Astronomer Royal :—

ELEMENTARY TREATISE ON PARTIAL DIFFERENTIAL EQUATIONS. Designed for the Use of Students in the Universities. With Diagrams. Crown 8vo. cloth. 5*s.* 6*d.*

It is hoped that the methods of solution here explained, and the instances exhibited, will be found sufficient for application to nearly all the important problems of Physical Science, which require for their complete investigation the aid of Partial Differential Equations.

ON THE ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS OF OBSERVATIONS AND THE COMBINATION OF OBSERVATIONS. Crown 8vo. cloth. 6*s.* 6*d.*

In order to spare astronomers and observers in natural philosophy the confusion and loss of time which are produced by referring to the ordinary treatises embracing both *branches of probabilities (the first relating to chances which can be altered only by the changes of entire units or integral multiples of units in the fundamental conditions of the problem ; the other concerning those chances which have respect to insensible gradations in the value of the element measured) the present tract has been drawn up. It relates only to errors of observation, and to the rules, derivable from the consideration of these errors, for the combination of the results of observations.*

Airy (G. B.)—*continued.*

UNDULATORY THEORY OF OPTICS. Designed for the Use ot Students in the University. New Edition. Crown 8vo. cloth. 6*s*. 6*d*.

The undulatory theory of optics is presented to the reader as having the same claims to his attention as the theory of gravitation: namely, that it is certainly true, and that, by mathematical operations of general elegance, it leads to results of great interest. This theory explains with accuracy a vast variety of phenomena of the most complicated kind. The plan of this tract has been to include those phenomena only which admit of calculation, and the investigations are applied only to phenomena which actually have been observed.

ON SOUND AND ATMOSPHERIC VIBRATIONS. With the Mathematical Elements of Music. Designed for the Use of Students of the University. Second Edition, Revised and Enlarged. Crown 8vo. 9*s*.

This volume consists of sections, which again are divided into numbered articles, on the following topics: General recognition of the air as the medium which conveys sound; Properties of the air on which the formation and transmission of sound depend; Theory of undulations as applied to sound, &c.; Investigation of the motion of a wave of air through the atmosphere; Transmission of waves of soniferous vibrations through different gases, solids, and fluids; Experiments on the velocity of sound, &c.; On musical sounds, and the manner of producing them; On the elements of musical harmony and melody, and of simple musical composition; On instrumental music; On the human organs of speech and hearing.

A TREATISE ON MAGNETISM. Designed for the use of Students in the University. Crown 8vo. 9*s*. 6*d*.

As the laws of Magnetic Force have been experimentally examined with philosophical accuracy, only in its connection with iron and steel, and in *the influences excited by the earth as a whole, the accurate portions of this work are confined to the investigations connected with these metals and the earth. The latter part of the work, however, treats in a more general way of the laws of the connection between Magnetism on the one hand and galvanism and thermo-electricity on the other. The work is divided into Twelve Sections, and each section into numbered articles, each of which states concisely and clearly the subject of the following paragraphs.*

Airy (Osmund.) — A TREATISE ON GEOMETRICAL OPTICS. Adapted for the use of the Higher Classes in Schools. By OSMUND AIRY, B.A., one of the Mathematical Masters in Wellington College. Extra fcap. 8vo. 3*s.* 6*d.*

"*This is, I imagine, the first time that any attempt has been made to adapt the subject of Geometrical Optics, to the reading of the higher classes in our good schools. That this should be so is the more a matter for remark, since the subject would appear to be peculiarly fitted for such an adaptation. . . . . I have endeavoured, as much as possible, to avoid the example of those popular lecturers who explain difficulties by ignoring them. But as the nature of my design necessitated brevity, I have omitted entirely one or two portions of the subject which I considered unnecessary to a clear understanding of the rest, and which appear to me better learnt at a more advanced stage.*"—AUTHOR'S PREFACE. "*This book,*" the ATHENÆUM *says,* "*is carefully and lucidly written, and rendered as simple as possible by the use in all cases of the most elementary form of investigation.*"

Bayma.—THE ELEMENTS OF MOLECULAR MECHANICS. By JOSEPH BAYMA, S. J., Professor of Philosophy, Stonyhurst College. Demy 8vo. cloth. 10*s.* 6*d.*

Of the twelve Books into which the present treatise is divided, the first and second give the demonstration of the principles which bear directly on the constitution and the properties of matter. The next three books contain a series of theorems and of problems on the laws of motion of elementary substances. In the sixth and seventh, the mechanical constitution of molecules is investigated and determined: and by it the general properties of bodies are explained. The eighth book treats of luminiferous æther. The *ninth explains some special properties of bodies. The tenth and eleventh contain a radical and lengthy investigation of chemical principles and relations, which may lead to practical results of high importance. The twelfth and last book treats of molecular masses, distances, and powers.*

Beasley.—AN ELEMENTARY TREATISE ON PLANE TRIGONOMETRY. With Examples. By R. D. BEASLEY, M.A., Head Master of Grantham Grammar School. Second Edition, revised and enlarged. Crown 8vo. cloth. 3*s.* 6*d.*

This treatise is specially intended for use in schools. The choice of matter has been chiefly guided by the requirements of the three days' examination at Cambridge. About four hundred examples have been added to this edition, mainly collected from the Examination Papers of the last ten years.

Blackburn (Hugh.)—ELEMENTS OF PLANE TRIGONOMETRY for the use of the Junior Class of Mathematics in the University of Glasgow. By HUGH BLACKBURN, M.A., Professor of Mathematics in the University of Glasgow. Globe 8vo. 1*s.* 6*d.*

The author having felt the want of a short treatise to be used as a Text-Book after the Sixth Book of Euclid had been learned and some knowledge of Alegebra acquired, which should contain satisfactory demonstrations of the propositions to be used in teaching Junior Students the solution of Triangles, and should at the same time lay a solid foundation for the study of Analytical Trigonometry, thinking that others may have felt the same want, has attempted to supply it by the publication of this little work.

Boole.—Works by G. BOOLE, D.C.L., F.R.S., Professor of Mathematics in the Queen's University, Ireland.

A TREATISE ON DIFFERENTIAL EQUATIONS. New and Revised Edition. Edited by I. TODHUNTER. Crown 8vo. cloth. 14*s.*

Professor Boole has endeavoured in this treatise to convey as complete an account of the present state of knowledge on the subject of Differential Equations, as was consistent with the idea of a work intended, primarily, for

Boole—*continued.*

elementary instruction. The earlier sections of each chapter contain that kind of matter which has usually been thought suitable for the beginner, while the latter ones are devoted either to an account of recent discovery, or the discussion of such deeper questions of principle as are likely to present themselves to the reflective student in connexion with the methods and processes of his previous course. "A treatise incomparably superior to any other elementary book on the same subject with which we are acquainted."—PHILOSOPHICAL MAGAZINE.

A TREATISE ON DIFFERENTIAL EQUATIONS. Supplementary Volume. Edited by I. TODHUNTER. Crown 8vo. cloth. 8*s*. 6*d*.

This volume contains all that Professor Boole wrote for the purpose of enlarging his treatise on Differential Equations.

THE CALCULUS OF FINITE DIFFERENCES. Crown 8vo. cloth. 10*s*. 6*d*.

In this exposition of the Calculus of Finite Differences, particular attention has been paid to the connexion of its methods with those of the Differential Calculus—a connexion which in some instances involves far more than a merely formal analogy. The work is in some measure designed as a sequel to Professor Boole's Treatise on Differential Equations. *"As an original book by one of the first mathematicians of the age, it is out of all comparison with the mere second-hand compilations which have hitherto been alone accessible to the student."*—PHILOSOPHICAL MAGAZINE.

CAMBRIDGE SENATE-HOUSE PROBLEMS AND RIDERS, WITH SOLUTIONS:—

1848-1851.—PROBLEMS. By FERRERS and JACKSON. 8vo. cloth. 15*s*. 6*d*.

1848-1851.—RIDERS. By JAMESON. 8vo. cloth. 7*s*. 6*d*.

1854.—PROBLEMS AND RIDERS. By WALTON and MACKENZIE. 8vo. cloth. 10*s*. 6*d*.

1857.—PROBLEMS AND RIDERS. By CAMPION and WALTON. 8vo. cloth. 8*s*. 6*d*.

Boole—*continued.*

1860.—PROBLEMS AND RIDERS. By WATSON and ROUTH. Crown 8vo. cloth. 7*s.* 6*d.*

1864.—PROBLEMS AND RIDERS. By WALTON and WILKINSON. 8vo. cloth. 10*s.* 6*d.*

These volumes will be found of great value to Teachers and Students, as indicating the style and range of mathematical study in the University of Cambridge.

CAMBRIDGE COURSE OF ELEMENTARY NATURAL PHILOSOPHY, for the Degree of B. A. Originally compiled by J. C. SNOWBALL, M.A., late Fellow of St. John's College. Fifth Edition, revised and enlarged, and adapted for the Middle-Class Examinations by THOMAS LUND, B.D., Late Fellow and Lecturer of St. John's College, Editor of Wood's Algebra, &c. Crown 8vo. cloth. 5*s.*

This work will be found adapted to the wants, not only of University Students, but also of many others who require a short course of Mechanics and Hydrostatics, and especially of the candidates at our Middle Class Examinations. At the end of each chapter a series of easy questions is added for the exercise of the student.

CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL. The Complete Work, in Nine Vols. 8vo. cloth, 7*l.* 4*s.*

Only a few copies remain on hand. Among Contributors to this work will be found Sir W. Thomson, Stokes, Adams, Boole, Sir W. R. Hamilton, De Morgan, Cayley, Sylvester, Jellett, and other distinguished mathematicians.

Candler.—HELP TO ARITHMETIC. Designed for the use of Schools. By H. CANDLER, M.A. Mathematical Master of Uppingham School. Extra fcap. 8vo. 2*s.* 6*d.*

This work is intended as a companion to any text-book that may be in use. "The main difficulties which boys experience in the different rules are skilfully dealt with and removed."—MUSEUM.

Cheyne.—Works by C. H. H. CHEYNE, M.A., F.R.A.S.

AN ELEMENTARY TREATISE ON THE PLANETARY THEORY. With a Collection of Problems. Second Edition. Crown 8vo. cloth. 6*s.* 6*d.*

In this volume, an attempt has been made to produce a treatise on the Planetary theory, which, being elementary in character, should be so far complete, as to contain all that is usually required by students in the University of Cambridge. In the New Edition the work has been carefully revised. The stability of the Planetary System has been more fully treated, and an elegant geometrical explanation of the formulæ for the secular variation of the node and inclination has been introduced.

THE EARTH'S MOTION OF ROTATION. Crown 8vo. 3*s.* 6*d.*

The first part of this work consists of an application of the method of the variation of elements to the general problem of rotation. In the second part the general rotation formulæ are applied to the particular case of the earth.

Childe.—THE SINGULAR PROPERTIES OF THE ELLIPSOID AND ASSOCIATED SURFACES OF THE NTH DEGREE. By the Rev. G. F. CHILDE, M.A., Author of "Ray Surfaces," "Related Caustics," &c. 8vo. 10*s.* 6*d.*

The object of this volume is to develop peculiarities in the Ellipsoid; and, further, to establish analogous properties in the unlimited congeneric series of which this remarkable surface is a constituent.

Christie.—A COLLECTION OF ELEMENTARY TEST-QUESTIONS IN PURE AND MIXED MATHEMATICS; with Answers and Appendices on Synthetic Division, and on the Solution of Numerical Equations by Horner's Method. By JAMES R. CHRISTIE, F.R.S., late First Mathematical Master at the Royal Military Academy, Woolwich. Crown 8vo. cloth. 8*s.* 6*d.*

The series of Mathematical exercises here offered to the public is collected from those which the author has, from time to time, proposed for solution *by his pupils during a long career at the Royal Military Academy. A student who finds that he is able to solve the larger portion of these exercises, may consider that he is thoroughly well grounded in the elementary principles of pure and mixed Mathematics.*

Dalton.—ARITHMETICAL EXAMPLES. Progressively arranged, with Exercises and Examination Papers. By the Rev. T. DALTON, M.A., Assistant Master of Eton College. 18mo. cloth. 2*s.* 6*d.* *Answers to the Examples are appended.*

Day.—PROPERTIES OF CONIC SECTIONS PROVED GEOMETRICALLY. PART I., THE ELLIPSE, with Problems. By the Rev. H. G. DAY, M.A., Head Master of Sedburgh Grammar School. Crown 8vo. 3*s.* 6*d.*

The object of this book is the introduction of a treatment of Conic Sections which should be simple and natural, and lead by an easy transition to the analytical methods, without departing from the strict geometry of Euclid.

Dodgson.—AN ELEMENTARY TREATISE ON DETERMINANTS, with their Application to Simultaneous Linear Equations and Algebraical Geometry. By CHARLES L DODGSON, M.A., Student and Mathematical Lecturer of Christ Church, Oxford. Small 4to. cloth. 10*s.* 6*d.*

The object of the author is to present the subject as a continuous chain of argument, separated from all accessories of explanation or illustration. All such explanation and illustration as seemed necessary for a beginner are introduced either in the form of foot-notes, or, where that would have occupied too much room, of Appendices. "The work," says the EDUCATIONAL TIMES, *"forms a valuable addition to the treatises we possess on modern Algebra."*

Drew.—GEOMETRICAL TREATISE ON CONIC SECTIONS. By W. H. DREW, M.A., St. John's College, Cambridge. Fourth Edition. Crown 8vo. cloth. 4*s.* 6*d.*

In this work the subject of Conic Sections has been placed before the student

Drew—*continued.*

in such a form that, it is hoped, after mastering the elements of Euclid, he may find it an easy and interesting continuation of his geometrical studies. With a view, also, of rendering the work a complete manual of what is required at the Universities, there have either been embodied into the text or inserted among the examples, every book-work question, problem, and rider, which has been proposed in the Cambridge examinations up to the present time.

SOLUTIONS TO THE PROBLEMS IN DREW'S CONIC SECTIONS. Crown 8vo. cloth. 4*s*. 6*d*.

Earnshaw (S.) — PARTIAL DIFFERENTIAL EQUATIONS. An Essay towards an entirely New Method of Integrating them. By S. EARNSHAW, M.A., St. John's College, Cambridge. Crown 8vo. 5*s*.

Edgar (J. H.) and Pritchard (G. S.)—NOTE-BOOK ON PRACTICAL SOLID OR DESCRIPTIVE GEOMETRY. Containing Problems with help for Solutions. By J. H. EDGAR, M.A., Lecturer on Mechanical Drawing at the Royal School of Mines, and G. S. PRITCHARD, late Master for Descriptive Geometry, Royal Military Academy, Woolwich. Second Edition, revised and enlarged. Globe 8vo. 3*s*.

Ferrers.—AN ELEMENTARY TREATISE ON TRILINEAR CO-ORDINATES, the Method of Reciprocal Polars, and the Theory of Projectors. By the Rev. N. M. FERRERS, M.A., Fellow and Tutor of Gonville and Caius College, Cambridge. Second Edition. Crown 8vo. 6*s*. 6*d*.

The object of the author in writing on this subject has mainly been to place it on a basis altogether independent of the ordinary Cartesian system, instead of regarding it as only a special form of Abridged Notation. A short chapter on Determinants has been introduced.

Frost.—THE FIRST THREE SECTIONS OF NEWTON'S PRINCIPIA. With Notes and Illustrations. Also a collection of Problems, principally intended as Examples of Newton's Methods. By PERCIVAL FROST, M.A., late Fellow of St. John's College, Mathematical Lecturer of King's College, Cambridge. Second Edition. 8vo. cloth. 10*s*. 6*d*.

The author's principal intention is to explain difficulties which may be encountered by the student on first reading the Principia, *and to illustrate the advantages of a careful study of the methods employed by Newton, by showing the extent to which they may be applied in the solution of problems; he has also endeavoured to give assistance to the student who is engaged in the study of the higher branches of mathematics, by representing in a geometrical form several of the processes employed in the Differential and Integral Calculus, and in the analytical investigations of Dynamics.*

Frost and Wolstenholme.—A TREATISE ON SOLID GEOMETRY. By PERCIVAL FROST, M.A., and the Rev. J. WOLSTENHOLME, M.A., Fellow and Assistant Tutor of Christ's College. 8vo. cloth. 18*s*.

The authors have endeavoured to present before students as comprehensive a view of the subject as possible. Intending to make the subject accessible, at least in the earlier portion, to all classes of students, they have endeavoured to explain completely all the processes which are most useful in dealing with ordinary theorems and problems, thus directing the student to the selection of methods which are best adapted to the exigencies of each problem. In the more difficult portions of the subject, they have considered themselves to be addressing a higher class of students; and they have there tried to lay a good foundation on which to build, if any reader should wish to pursue the science beyond the limits to which the work extends.

Godfray.—Works by HUGH GODFRAY, M.A. Mathematical Lecturer at Pembroke College, Cambridge.

A TREATISE ON ASTRONOMY, for the Use of Colleges and Schools. 8vo. cloth. 12*s*. 6*d*.

This book embraces all those branches of Astronomy which have, from time to time, been recommended by the Cambridge Board of Mathematical

Godfray—*continued.*

Studies: but by far the larger and easier portion, adapted to the first three days of the Examination for Honours, may be read by the more advanced pupils in many of our schools. The author's aim has been to convey clear and distinct ideas of the celestial phenomena. "It is a working book," says the GUARDIAN, *"taking Astronomy in its proper place in mathematical sciences. . . . It is a book which is not likely to be got up unintelligently."*

AN ELEMENTARY TREATISE ON THE LUNAR THEORY, with a Brief Sketch of the Problem up to the time of Newton. Second Edition, revised. Crown 8vo. cloth. 5*s.* 6*d.*

These pages will, it is hoped, form an introduction to more recondite works. Difficulties have been discussed at considerable length. The selection of the method followed with regard to analytical solutions, which is the same as that of Airy, Herschel, &c. was made on account of its simplicity; it is, moreover, the method which has obtained in the University of Cambridge. "As an elementary treatise and introduction to the subject, we think it may justly claim to supersede all former ones."—LONDON, EDIN. AND DUBLIN PHIL. MAGAZINE.

Hemming.—AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS, for the Use of Colleges and Schools. By G. W. HEMMING, M.A., Fellow of St. John's College, Cambridge. Second Edition, with Corrections and Additions. 8vo. cloth. 9*s.*

"There is no book in common use from which so clear and exact a knowledge of the principles of the Calculus can be so readily obtained."—LITERARY GAZETTE.

Jones and Cheyne.—ALGEBRAICAL EXERCISES. Progressively arranged. By the Rev. C. A. JONES, M.A., and C. H. CHEYNE, M.A., F.R.A.S., Mathematical Masters of Westminster School. New Edition. 18mo. cloth. 2*s.* 6*d.*

This little book is intended to meet a difficulty which is probably felt more or less by all engaged in teaching Algebra to beginners. It is, that while *new ideas are being acquired, old ones are forgotten. In the belief that constant practice is the only remedy for this, the present series of miscellaneous exercises has been prepared. Their peculiarity consists in this, that though miscellaneous they are yet progressive, and may be used by the pupil almost from the commencement of his studies. They are not intended to supersede the systematically arranged examples to be found in ordinary treatises on Algebra, but rather to supplement them. The book being intended chiefly for Schools and Junior Students, the higher parts of Algebra have not been included.*

Kitchener.—A GEOMETRICAL NOTE-BOOK, containing Easy Problems in Geometrical Drawing preparatory to the Study of Geometry. For the Use of Schools. By F. E. KITCHENER, M.A., Mathematical Master at Rugby. 4to. 2*s*.

It is the object of this book to make some way in overcoming the difficulties of Geometrical conception, before the mind is called to the attack of Geometrical theorems. A few simple methods of construction are given; and space is left on each page, in order that the learner may draw in the figures.

Morgan.—A COLLECTION OF PROBLEMS AND EXAMPLES IN MATHEMATICS. With Answers. By H. A. MORGAN, M.A., Sadlerian and Mathematical Lecturer of Jesus College, Cambridge. Crown 8vo. cloth. 6*s*. 6*d*.

This book contains a number of problems, chiefly elementary, in the Mathematical subjects usually read at Cambridge. They have been selected from the papers set during late years at Jesus College. Very few of them are to be met with in other collections, and by far the larger number are due to some of the most distinguished Mathematicians in the University.

Newton's PRINCIPIA. 4to. cloth. 31*s*. 6*d*.

It is a sufficient guarantee of the reliability of this complete edition of Newton's Principia *that it has been printed for and under the care of Professor Sir William Thomson and Professor Blackburn, of Glasgow University. The following notice is prefixed:—" Finding that all the editions* *of the* Principia *are now out of print, we have been induced to reprint Newton's last edition* [*of* 1726] *without note or comment, only introducing the 'Corrigenda' of the old copy and correcting typographical errors." The book is of a handsome size, with large type, fine thick paper, and cleanly cut figures, and is the only modern edition containing the whole of Newton's great work.*

Parkinson.—Works by S. PARKINSON, D.D., F.R.S., Fellow and Tutor of St. John's College, Cambridge.

AN ELEMENTARY TREATISE ON MECHANICS. For the Use of the Junior Classes at the University and the Higher Classes in Schools. With a Collection of Examples. Fourth edition, revised. Crown 8vo. cloth. 9*s.* 6*d.*

In preparing a fourth edition of this work the author has kept the same object in view as he had in the former editions—namely, to include in it such portions of Theoretical Mechanics as can be conveniently investigated without the use of the Differential Calculus, and so render it suitable as a manual for the junior classes in the University and the higher classes in Schools. With one or two short exceptions, the student is not presumed to require a knowledge of any branches of Mathematics beyond the elements of Algebra, Geometry, and Trigonometry. Several additional propositions have been incorporated in the work for the purpose of rendering it more complete; and the collection of Examples and Problems has been largely increased.

A TREATISE ON OPTICS. Third Edition, revised and enlarged. Crown 8vo. cloth. 10*s.* 6*d.*

A collection of examples and problems has been appended to this work, which are sufficiently numerous and varied in character to afford useful exercise for the student. For the greater part of them, recourse has been had to the Examination Papers set in the University and the several Colleges during the last twenty years.

Phear.—ELEMENTARY HYDROSTATICS. With Numerous Examples. By J. B. PHEAR, M.A., Fellow and late Assistant Tutor of Clare College, Cambridge. Fourth Edition. Crown 8vo. cloth. 5*s.* 6*d.*

This edition has been carefully revised throughout, and many new illustrations and examples added, which it is hoped will increase its usefulness to students at the Universities and in Schools. In accordance with suggestions from many engaged in tuition, answers to all the Examples have been given at the end of the book.

Pratt.—A TREATISE ON ATTRACTIONS, LAPLACE'S FUNCTIONS, AND THE FIGURE OF THE EARTH. By JOHN H. PRATT, M.A., Archdeacon of Calcutta, Author of "The Mathematical Principles of Mechanical Philosophy." Third Edition. Crown 8vo. cloth. 6*s.* 6*d.*

The author's chief design in this treatise is to give an answer to the question, "Has the Earth acquired its present form from being originally in a fluid state?" This Edition is a complete revision of the former ones.

Puckle.—AN ELEMENTARY TREATISE ON CONIC SECTIONS AND ALGEBRAIC GEOMETRY. With Numerous Examples and Hints for their Solution; especially designed for the Use of Beginners. By G. H. PUCKLE, M.A., Head Master of Windermere College. New Edition, revised and enlarged. Crown 8vo. cloth. 7*s.* 6*d.*

This work is recommended by the Syndicate of the Cambridge Local Examinations, and is the text-book in Harvard University, U.S. The ATHENÆUM *says the Author "displays an intimate acquaintance with the difficulties likely to be felt, together with a singular aptitude in removing them."*

Rawlinson.—ELEMENTARY STATICS, by the Rev. GEORGE RAWLINSON, M.A. Edited by the Rev. EDWARD STURGES, M.A., of Emmanuel College, Cambridge, and late Professor of the Applied Sciences, Elphinstone College, Bombay. Crown 8vo. cloth. 4*s.* 6*d.*

Published under the authority of Her Majesty's Secretary of State for India, for use in the Government Schools and Colleges in India. "This manual may take its place among the most exhaustive, yet clear and simple, we have met with."—ORIENTAL BUDGET.

Reynolds.—MODERN METHODS IN ELEMENTARY GEOMETRY. By E. M. REYNOLDS, M.A., Mathematical Master in Clifton College. Crown 8vo. 3*s.* 6*d.*

Some change, it is evident, in our English ways of teaching can now no longer be postponed, and this little book, mainly derived from French and German sources, has been written in the hope of facilitating that change. It has been constructed on one plan throughout, that of always giving in the simplest possible form the direct proof from the nature of the case. The axioms necessary to this simplicity have been assumed without hesitation, and no scruple has been felt as to the increase of their number, or the acceptance of as many elementary notions as common experience places past all doubt. The book differs most from established teaching in its constructions, and in its early application of Arithmetic to Geometry.

Routh.—AN ELEMENTARY TREATISE ON THE DYNAMICS OF THE SYSTEM OF RIGID BODIES. With Numerous Examples. By EDWARD JOHN ROUTH, M.A., late Fellow and Assistant Tutor of St. Peter's College, Cambridge; Examiner in the University of London. Second Edition, enlarged. Crown 8vo. cloth. 14*s.*

In this edition the author has made several additions to each chapter. He has tried, even at the risk of some little repetition, to make each chapter, as far as possible, complete in itself, so that all that relates to any one part of the subject may be found in the same place. This arrangement will enable every student to select his own order in which to read the subject. The Examples which will be found at the end of each chapter have been chiefly selected from the Examination Papers which have been set in the University and the Colleges in the last few years.

Smith (Barnard).—Works by BARNARD SMITH, M.A., Rector of Glaston, Rutlandshire, late Fellow and Senior Bursar of St. Peter's College, Cambridge.

Smith (Barnard)—*continued.*

ARITHMETIC AND ALGEBRA, in their Principles and Application; with numerous systematically arranged Examples taken from the Cambridge Examination Papers, with especial reference to the Ordinary Examination for the B.A. Degree. Eleventh Edition. Crown 8vo. cloth. 10*s.* 6*d.*

This manual is now extensively used in Schools and Colleges, both in England and in the Colonies. It has also been found of great service for students preparing for the Middle Class and Civil and Military Service Examinations, from the care that has been taken to elucidate the principles *of all the rules. The present edition has been carefully revised. "To all those whose minds are sufficiently developed to comprehend the simplest mathematical reasoning, and who have not yet thoroughly mastered the principles of Arithmetic and Algebra, it is calculated to be of great advantage."*—ATHENÆUM. *Of this work, also, one of the highest possible authorities, the late Dean Peacock, writes: "Mr. Smith's work is a most useful publication. The rules are stated with great clearness. The examples are well selected, and worked out with just sufficient detail, without being encumbered by too minute explanations; and there prevails throughout it that just proportion of theory and practice, which is the crowning excellence of an elementary work."*

ARITHMETIC FOR SCHOOLS. New Edition. Crown 8vo. cloth. 4*s.* 6*d.*

Adapted from the author's work on "Arithmetic and Algebra," by the omission of the algebraic portion, and by the introduction of new exercises. The reason of each arithmetical process is fully exhibited. The system of Decimal Coinage is explained; and answers to the exercises are appended at the end. This Arithmetic is characterised as "admirably adapted for instruction, combining just sufficient theory with a large and well-selected collection of exercises for practice."—JOURNAL OF EDUCATION.

COMPANION TO ARITHMETIC FOR SCHOOLS. [*Preparing.*

Smith (Barnard)—*continued.*

A KEY TO THE ARITHMETIC FOR SCHOOLS. Seventh Edition. Crown 8vo. cloth. 8*s.* 6*d.*

EXERCISES IN ARITHMETIC. With Answers. Crown 8vo. limp cloth. 2*s.* 6*d.*

Or sold separately, Part I. 1*s.*; Part II. 1*s.*; Answers, 6*d.*

These Exercises have been published in order to give the pupil examples in every rule of Arithmetic. The greater number have been carefully compiled from the latest University and School Examination Papers.

SCHOOL CLASS-BOOK OF ARITHMETIC. 18mo. cloth. 3*s.* Or sold separately, Parts I. and II. 10*d.* each; Part III. 1*s.*

This manual, published at the request of many schoolmasters, and chiefly intended for National and Elementary Schools, has been prepared on the same plan as that adopted in the author's School Arithmetic, which is in extensive circulation in England and abroad. The Metrical Tables have been introduced, from the conviction on the part of the author, that the knowledge of such tables, and the mode of applying them, will be o great use to the rising generation.

KEYS TO SCHOOL CLASS-BOOK OF ARITHMETIC. Complete in one volume, 18mo. cloth, 6*s.* 6*d.*; or Parts I. II. and III. 2*s.* 6*d.* each.

SHILLING BOOK OF ARITHMETIC FOR NATIONAL AND ELEMENTARY SCHOOLS. 18mo. cloth. Or separately, Part I. 2*d.*; Part II. 3*d.*; Part III. 7*d.* Answers, 6*d.*

THE SAME, with Answers complete. 18mo. cloth. 1*s.* 6*d.*

This Shilling Book of Arithmetic has been prepared for the use of National and other schools at the urgent request of numerous masters of schools both at home and abroad. The Explanations of the Rules, and the Examples will, it is hoped, be found suited to the most elementary classes.

Smith (Barnard)—*continued.*

KEY TO SHILLING BOOK OF ARITHMETIC. 18mo. cloth. 4*s*. 6*d*.

EXAMINATION PAPERS IN ARITHMETIC. 18mo. cloth. 1*s*. 6*d*. The same, with Answers, 18mo. 1*s*. 9*d*.

The object of these Examination Papers is to test students both in the theory and practice of Arithmetic. It is hoped that the method adopted will lead students to deduce results from general principles rather than to apply stated rules. The author believes that the practice of giving examples under particular rules makes the working of Arithmetic quite mechanical, and tends to throw all but very clever boys off their balance when a general paper on the subject is put before them.

KEY TO EXAMINATION PAPERS IN ARITHMETIC. 18mo. cloth. 4*s*. 6*d*.

THE METRIC SYSTEM OF ARITHMETIC, ITS PRINCIPLES AND APPLICATION, with numerous Examples, written expressly for Standard V. in National Schools. New Edition. 18mo. cloth sewed. 3*d*.

In the New Code of Regulations issued by the Council of Education it is stated "that in all schools children in Standards V. and VI. should know the principles of the Metric System, and be able to explain the advantages to be gained from uniformity in the method of forming multiples and sub-multiples of the unit." In this little book Mr. Smith in a clear, simple, and interesting manner explains the principle of the Metric System, and in considerable detail expounds the French system, and its relation to the ordinary English method, taking the pupil on as far as Compound Division. The book contains numerous Examples, and two wood-cuts illustrating the Metric Tables of Surface and Solidity. Answers to the Examples are appended.

A CHART OF THE METRIC SYSTEM, on a Sheet, size 42 in. by 34 in. on Roller, 1*s*. 6*d*. On Roller, mounted and varnished, price 3*s*. 6*d*. Also on a Card, price 1*d*.

By the New Educational Code it is ordained that a Chart of the Metric System be conspicuously hung up on the walls of every school under government inspection. The publishers believe that the present Chart will be found to answer all the requirements of the Code, and afford a full and perfectly intelligible view of the principles of the Metric System. The principle of the system is clearly stated and illustrated by examples; the Method of Forming the Tables is set forth; Tables follow, clearly shewing the English equivalent of the French measures of—1. *Length;* 2. *Surface;* 3. *Solidity;* 4. *Weight;* 5. *Capacity. At the bottom of the Chart is drawn a full-length Metric Measure, subdivided distinctly and intelligently into Decimetres, Centimetres, and Millimetres.*

Smith (J. Brook).—ARITHMETIC IN THEORY AND PRACTICE, FOR ADVANCED PUPILS. By J. BROOK SMITH, M.A. Part I. Crown 8vo. 3*s.* 6*d.*

The following pages form the first part of a Treatise on Arithmetic, in which the Author has endeavoured from very simple principles to explain, in a full and satisfactory manner, all the more important processes in that subject. The proofs have in all cases been given in a form entirely arithmetical, and at the end of every chapter several examples have been worked out at length, and the best practical method of operation carefully pointed out.

Snowball.—THE ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY; with the Construction and Use of Tables of Logarithms. By J. C. SNOWBALL, M.A. Tenth Edition. Crown 8vo. cloth. 7*s.* 6*d.*

In preparing the present edition for the press, the text has been subjected to a careful revision; the proofs of some of the more important propositions have been rendered more strict and general; and a considerable addition of more than two hundred examples, taken principally from the questions set of late years in the public examinations of the University and of individual Colleges, has been made to the collection of Examples and Problems for practice.

Tait and Steele.—A TREATISE ON DYNAMICS OF A PARTICLE. With numerous Examples. By Professor TAIT and Mr. STEELE. New Edition Enlarged. Crown 8vo. cloth. 10*s.* 6*d.*

In this treatise will be found all the ordinary propositions, connected with the Dynamics of Particles, which can be conveniently deduced without the use of D'Alembert's Principle. Throughout the book will be found a number of illustrative examples introduced in the text, and for the most part completely worked out; others with occasional solutions or hints to assist the student are appended to each chapter. For by far the greater portion of these, the Cambridge Senate-House and College Examination Papers have been applied to. In the new edition numerous trivial errors, and a few of a more serious character, have been corrected, while many new examples have been added.

Taylor.—GEOMETRICAL CONICS; including Anharmonic Ratio and Projection, with numerous Examples. By C. TAYLOR, B.A., Scholar of St. John's College, Cambridge. Crown 8vo. cloth. 7*s.* 6*d.*

This work contains elementary proofs of the principal properties of Conic Sections, together with chapters on Projection and Anharmonic Ratio.

Tebay.—ELEMENTARY MENSURATION FOR SCHOOLS. With numerous Examples. By SEPTIMUS TEBAY, B.A., Head Master of Queen Elizabeth's Grammar School, Rivington. Extra fcap. 8vo. 3*s.* 6*d.*

The object of the present work is to enable boys to acquire a moderate knowledge of Mensuration in a reasonable time. All difficult and useless matter has been avoided. The examples for the most part are easy, and the rules are concise. "A very compact useful manual."—SPECTATOR.

Todhunter.—Works by I. TODHUNTER, M.A., F.R.S., of St. John's College, Cambridge.

"They are all good, and each volume adds to the value of the rest."—FREEMAN. *"Perspicuous language, vigorous investigations, scrutiny of*

Todhunter (I.)—*continued.*

difficulties, and methodical treatment, characterise Mr. Todhunter's works."—CIVIL ENGINEER.

THE ELEMENTS OF EUCLID. For the Use of Colleges and Schools. New Edition. 18mo. cloth. 3*s.* 6*d.*

As the elements of Euclid are usually placed in the hands of young students, it is important to exhibit the work in such a form as will assist them in overcoming the difficulties which they experience on their first introduction to processes of continuous argument. No method appears to be so useful as that of breaking up the demonstrations into their constituent parts; a plan strongly recommended by Professor De Morgan. In the present Edition each distinct assertion in the argument begins a new line: and at the ends of the lines are placed the necessary references to the preceding principles on which the assertions depend. The longer propositions are distributed into subordinate parts, which are distinguished by breaks at the beginning of the lines. Notes, appendix, and a collection of exercises are added.

MENSURATION FOR BEGINNERS. With Numerous Examples. 18mo. cloth. 2*s.* 6*d.*

The subjects included in the present work are those which have usually found a place in Elementary Treatises on Mensuration. The mode of treatment has been determined by the fact that the work is intended for the use of beginners. Accordingly it is divided into short independent chapters, which are followed by appropriate examples. A knowledge of the elements of Arithmetic is all that is assumed; and in connexion with most of the Rules of Mensuration it has been found practicable to give such explanations and illustrations as will supply the place of formal mathematical demonstrations, which would have been unsuitable to the character of the work. "For simplicity and clearness of arrangement it is unsurpassed by any text-book on the subject which has come under our notice."—EDUCATIONAL TIMES.

Todhunter (I.)—*continued.*

ALGEBRA FOR BEGINNERS. With numerous Examples. New Edition. 18mo. cloth. 2*s*. 6*d*.

Great pains have been taken to render this work intelligible to young students, by the use of simple language and by copious explanations. In determining the subjects to be included and the space to be assigned to each, the Author has been guided by the papers given at the various examinations in elementary Algebra which are now carried on in this country. The book may be said to consist of three parts. The first part contains the elementary operations in integral and fractional expressions; the second the solution of equations and problems; the third treats of various subjects which are introduced but rarely into examination papers, and are more briefly discussed. Provision has at the same time been made for the introduction of easy equations and problems at an early stage—for those who prefer such a course.

KEY TO ALGEBRA FOR BEGINNERS. Crown 8vo. cloth. 6*s*. 6*d*.

TRIGONOMETRY FOR BEGINNERS. With numerous Examples. New Edition. 18mo. cloth. 2*s*. 6*d*.

Intended to serve as an introduction to the larger treatise on Plane Trigonometry, published by the Author. The same plan has been adopted as in the Algebra for Beginners: *the subject is discussed in short chapters, and a collection of examples is attached to each chapter. The first fourteen chapters present the geometrical part of Plane Trigonometry; and contain all that is necessary for practical purposes. The range of matter included is such as seems required by the various examinations in elementary Trigonometry which are now carried on in the country. Answers are appended.*

MECHANICS FOR BEGINNERS. With numerous Examples. Second Edition. 18mo. cloth. 4*s*. 6*d*.

Intended as a companion to the two preceding books. The work forms an elementary treatise on demonstrative mechanics. It may be true that this part of mixed mathematics has been sometimes made too abstract and speculative; but it can hardly be doubted that a knowledge of the elements

Todhunter (I.)—*continued.*

at least of the theory of the subject is extremely valuable even for thos who are mainly concerned with practical results. The Author has accordingly endeavoured to provide a suitable introduction to the study of appliea as well as of theoretical mechanics. The work consists of two parts, namely, Statics and Dynamics. It will be found to contain all that is usually comprised in elementary treatises on Mechanics, together with some additions.

ALGEBRA. For the Use of Colleges and Schools. Fifth Edition. Crown 8vo. cloth. 7*s*. 6*d*.

This work contains all the propositions which are usually included in elementary treatises on Algebra, and a large number of Examples for Exercise. *The author has sought to render the work easily intelligible to students, without impairing the accuracy of the demonstrations, or contracting the limits of the subject. The Examples, about* Sixteen hundred and fifty *in number, have been selected with a view to illustrate every part of the subject. Each chapter is complete in itself; and the work will be found peculiarly adapted to the wants of students who are without the aid of a teacher. The Answers to the examples, with hints for the solution of some in which assistance may be needed, are given at the end of the book. In the present edition two New Chapters and* Three hundred *miscellaneous Examples have been added. The latter are arranged in sets, each set containing ten examples. "It has merits which unquestionably places it first in the class to which it belongs."*—EDUCATOR.

KEY TO ALGEBRA FOR THE USE OF COLLEGES AND SCHOOLS. Crown 8vo. 10*s*. 6*d*.

AN ELEMENTARY TREATISE ON THE THEORY OF EQUATIONS. Second Edition, revised. Crown 8vo. cloth. 7*s*. 6*d*.

This treatise contains all the propositions which are usually included in elementary treatises on the theory of Equations, together with Examples for exercise. These have been selected from the College and University

Todhunter (I.)—*continued.*

Examination Papers, and the results have been given when it appeared necessary. In order to exhibit a comprehensive view of the subject, the treatise includes investigations which are not found in all the preceding elementary treatises, and also some investigations which are not to be found in any of them. For the second edition the work has been revised and some additions have been made, the most important being an account of the researches of Professor Sylvester respecting Newton's Rule. "A thoroughly trustworthy, complete, and yet not too elaborate treatise." PHILOSOPHICAL MAGAZINE.

PLANE TRIGONOMETRY. For Schools and Colleges. Fourth Edition. Crown 8vo. cloth. 5*s.*

The design of this work has been to render the subject intelligible to beginners, and at the same time to afford the student the opportunity of obtaining all the information which he will require on this branch of Mathematics. Each chapter is followed by a set of Examples: those which are entitled Miscellaneous Examples, *together with a few in some of the other sets, may be advantageously reserved by the student for exercise after he has made some progress in the subject. In the Second Edition the hints for the solution of the Examples have been considerably increased.*

A TREATISE ON SPHERICAL TRIGONOMETRY. Second Edition, enlarged. Crown 8vo. cloth. 4*s.* 6*d.*

The present work is constructed on the same plan as the treatise on Plane Trigonometry, to which it is intended as a sequel. In the account of Napier's Rules of Circular Parts, an explanation has been given of a method of proof devised by Napier, which seems to have been overlooked by most modern writers on the subject. Considerable labour has been bestowed on the text in order to render it comprehensive and accurate, and the Examples (selected chiefly from College Examination Papers) have all been carefully verified. "For educational purposes this work seems to be superior to any others on the subject."—CRITIC.

Todhunter (I.)—*continued.*

PLANE CO-ORDINATE GEOMETRY, as applied to the Straight Line and the Conic Sections. With numerous Examples. Fourth Edition, revised and enlarged. Crown 8vo. cloth. 7*s.* 6*d.*

The author has here endeavoured to exhibit the subject in a simple manner for the benefit of beginners, and at the same time to include in one volume all that students usually require. In addition, therefore, to the propositions which have always appeared in such treatises, he has introduced the methods of abridged notation, *which are of more recent origin; these methods, which are of a less elementary character than the rest of the work, are placed in separate chapters, and may be omitted by the student at first.*

A TREATISE ON THE DIFFERENTIAL CALCULUS. With numerous Examples. Fifth Edition. Crown 8vo. cloth. 10*s.* 6*d.*

The author has endeavoured in the present work to exhibit a comprehensive view of the Differential Calculus on the method of limits. In the more elementary portions he has entered into considerable detail in the explanations, with the hope that a reader who is without the assistance of a tutor may be enabled to acquire a competent acquaintance with the subject. The method adopted is that of Differential Coefficients. To the different chapters are appended examples sufficiently numerous to render another book unnecessary; these examples being mostly selected from College Examination Papers. "It has already taken its place as the *text-book on that subject."*—Philosophical Magazine.

A TREATISE ON THE INTEGRAL CALCULUS AND IT APPLICATIONS. With numerous Examples. Third Editio revised and enlarged. Crown 8vo. cloth. 10*s.* 6*d.*

This is designed as a work at once elementary and complete, adapted for the use of beginners, and sufficient for the wants of advanced students In the selection of the propositions, and in the mode of establishing them, it has been sought to exhibit the principles clearly, and to illustrate all their most important results. The process of summation has been

Todhunter (I.)—*continued.*

repeatedly brought forward, with the view of securing the attention of the student to the notions which form the true foundation of the Calculus itself, as well as of its most valuable applications. Every attempt has been made to explain those difficulties which usually perplex beginners, especially with reference to the limits *of integrations. A new method has been adopted in regard to the transformation of multiple integrals. The last chapter deals with the Calculus of Variations. A large collection of exercises, selected from College Examination Papers, has been appended to the several chapters.*

EXAMPLES OF ANALYTICAL GEOMETRY OF THREE DIMENSIONS. Second Edition, revised. Crown 8vo. cloth 4*s.*

A TREATISE ON ANALYTICAL STATICS. With numerous Examples. Third Edition, revised and enlarged. Crown 8vo. cloth. 10*s.* 6*d.*

In this work on statics (treating of the laws of the equilibrium of bodies) will be found all the propositions which usually appear in treatises on Theoretical Statics. To the different chapters examples are appended, which have been principally selected from University Examination Papers. In the Third Edition many additions have been made, in order to illustrate the application of the principles of the subject to the solution of problems.

Wilson (J. M.)—ELEMENTARY GEOMETRY. Angles, Parallels, Triangles, Equivalent Figures, the Circle, and Proportion. By J. M. WILSON, M.A., Fellow of St. John's College, Cambridge, and Mathematical Master in Rugby School. Second Edition. Extra fcap. 8vo. 3*s.* 6*d.*

The distinctive features of this work are intended to be the following. The classification of Theorems according to their subjects; the separation of Theorems and Problems; the use of hypothetical constructions; the adoption of independent proofs where they are possible and simple; the

Wilson (J. M.)—*continued.*

introduction of the terms locus, projection, &c. ; *the importance given to the notion of direction as the property of a straight line ; the intermixing of exercises, classified according to the methods adopted for their solution ; the diminution of the number of Theorems ; the compression of proofs, especially in the later parts of the book ; the tacit, instead of the explicit, reference to axioms ; and the treatment of parallels.* "*The methods employed have the great merit of suggesting a ready application to the solution of fresh problems.*"—GUARDIAN.

ELEMENTARY GEOMETRY. PART II. (separately). The Circle and Proportion. By J. M. WILSON, M.A. Extra fcap. 8vo. 2*s.* 6*d.*

Wilson (W. P.)—A TREATISE ON DYNAMICS. By W. P. WILSON, M.A., Fellow of St. John's College, Cambridge, and Professor of Mathematics in Queen's College, Belfast. 8vo. 9*s.* 6*d.*

Wolstenholme.—A BOOK OF MATHEMATICAL PROBLEMS, on Subjects included in the Cambridge Course. By JOSEPH WOLSTENHOLME, Fellow of Christ's College, sometime Fellow of St. John's College, and lately Lecturer in Mathematics at Christ's College. Crown 8vo. cloth. 8*s.* 6*d.*

CONTENTS:—*Geometry (Euclid)—Algebra—Plane Trigonometry—Geometrical Conic Sections—Analytical Conic Sections—Theory of Equations—Differential Calculus—Integral Calculus—Solid Geometry—Statics—Elementary Dynamics—Newton—Dynamics of a Point—Dynamics of a Rigid Body—Hydrostatics—Geometrical Optics—Spherical Trigonometry and Plane Astronomy.* "*Judicious, symmetrical, and well arranged.*"—GUARDIAN.

SCIENCE.

ELEMENTARY CLASS-BOOKS.

THE importance of Science as an element of sound education is now generally acknowledged; and accordingly it is obtaining a prominent place in the ordinary course of school instruction. It is the intention of the Publishers to produce a complete series of Scientific Manuals, affording full and accurate elementary information, conveyed in clear and lucid English. The authors are well known as among the foremost men of their several departments; and their names form a ready guarantee for the high character of the books. Subjoined is a list of those Manuals that have already appeared, with a short account of each. Others are in active preparation; and the whole will constitute a standard series specially adapted to the requirements of beginners, whether for private study or for school instruction.

ASTRONOMY, by the Astronomer Royal.

POPULAR ASTRONOMY. With Illustrations. By G. B. AIRY, Astronomer Royal. Sixth and cheaper Edition. 18mo. cloth. 4*s*. 6*d*.

This work consists of six lectures, which are intended "to explain to intelligent persons the principles on which the instruments of an Observatory are constructed (omitting all details, so far as they are merely sub-

Elementary Class-Books—*continued.*

sidiary), and the principles on which the observations made with these instruments are treated for deduction of the distances and weights of the bodies of the Solar System, and of a few stars, omitting all minutiæ of formulæ, and all troublesome details of calculation." The speciality of this volume is the direct reference of every step to the Observatory, and the full description of the methods and instruments of observation.

ASTRONOMY.

MR. LOCKYER'S ELEMENTARY LESSONS IN ASTRONOMY. With Coloured Diagram of the Spectra of the Sun, Stars, and Nebulæ, and numerous Illustrations. By J. NORMAN LOCKYER, F.R.S. Seventh Thousand. 18mo. 5*s.* 6*d.*

The author has here aimed to give a connected view of the whole subject, and to supply facts, and ideas founded on the facts, to serve as a basis for subsequent study and discussion. The chapters treat of the Stars and Nebulæ; the Sun; the Solar System; Apparent Movements of the Heavenly Bodies; the Measurement of Time; Light; the Telescope and Spectroscope; Apparent Places of the Heavenly Bodies; the Real Distances and Dimensions; Universal Gravitation. The most recent astronomical discoveries are incorporated. Mr. Lockyer's work supplements that of the Astronomer Royal mentioned in the previous article. "The book is full, clear, sound, and worthy of attention, not only as a popular exposition, but as a scientific 'Index.'"—ATHENÆUM. *"The most fascinating of elementary books on the Sciences."*—NONCONFORMIST.

QUESTIONS ON LOCKYER'S ELEMENTARY LESSONS IN ASTRONOMY. For the use of Schools. By JOHN FORBES-ROBERTSON. 18mo. cloth limp. 1*s.* 6*d.*

PHYSIOLOGY.

PROFESSOR HUXLEY'S LESSONS IN ELEMENTARY PHYSIOLOGY. With numerous Illustrations. By T. H. HUXLEY, F.R.S. Professor of Natural History in the Royal School of Mines. Seventeenth Thousand. 18mo. cloth. 4*s.* 6*d.*

Elementary Class-Books—*continued.*

This book describes and explains, in a series of graduated lessons, the principles of Human Physiology; or the Structure and Functions of the Human Body. The first lesson supplies a general view of the subject. This is followed by sections on the Vascular or Venous System, and the Circulation; the Blood and the Lymph; Respiration; Sources of Loss and of Gain to the Blood; the Function of Alimentation; Motion and Locomotion; Sensations and Sensory Organs; the Organ of Sight; the Coalescence of Sensations with one another and with other States of Consciousness; the Nervous System and Innervation; Histology, or the Minute Structure of the Tissues. A Table of Anatomical and Physiological Constants is appended. The lessons are fully illustrated by numerous engravings. The manual is primarily intended to serve as a text-book for teachers and learners in boys' and girls' schools. "Pure gold throughout."—GUARDIAN. "*Unquestionably the clearest and most complete elementary treatise on this subject that we possess in any language.*"—WESTMINSTER REVIEW.

QUESTIONS ON HUXLEY'S PHYSIOLOGY FOR SCHOOLS. By T. ALCOCK, M.D. 18mo. 1*s.* 6*d.*

These Questions were drawn up as aids to the instruction of a class of young people in Physiology.

BOTANY.

PROFESSOR OLIVER'S LESSONS IN ELEMENTARY BOTANY. With nearly Two Hundred Illustrations. Eleventh Thousand. 18mo. cloth. 4*s.* 6*d.*

This book is designed to teach the Elements of Botany on Professor Henslow's plan of selected Types and by the use of Schedules. The earlier chapters, embracing the elements of Structural and Physiological Botany, introduce us to the methodical study of the Ordinal Types. The concluding chapters are entitled, "How to dry Plants" and "How to describe Plants." A valuable Glossary is appended to the volume. In the preparation of this work free use has been made of the manuscript materials of the late Professor Henslow.

Elementary Class-Books—*continued.*

CHEMISTRY.

PROFESSOR ROSCOE'S LESSONS IN ELEMENTARY CHEMISTRY, INORGANIC AND ORGANIC. By HENRY E. ROSCOE, F.R.S., Professor of Chemistry in Owens College, Manchester. With numerous Illustrations and Chromo-Litho. of the Solar Spectrum, and of the Alkalies and Alkaline Earths, *New Edition.* Twenty-ninth Thousand. 18mo. cloth. 4*s.* 6*d.*

It has been the endeavour of the author to arrange the most important facts and principles of Modern Chemistry in a plain but concise and scientific form, suited to the present requirements of elementary instruction. For the purpose of facilitating the attainment of exactitude in the knowledge of the subject, a series of exercises and questions upon the lessons have been added. The metric system of weights and measures, and the centigrade thermometric scale, are used throughout the work. The new Edition, besides new wood-cuts, contains many additions and improvements, and includes the most important of the latest discoveries. "As a standard general text-book it deserves to take a leading place."—SPECTATOR. "*We unhesitatingly pronounce it the best of all our elementary treatises on Chemistry.*—MEDICAL TIMES.

POLITICAL ECONOMY.

POLITICAL ECONOMY FOR BEGINNERS. By MILLICENT G. FAWCETT. 18mo. 2*s.* 6*d.*

The following pages have been written mainly with the hope that a short and elementary book might help to make Political Economy a more popular study in boys' and girls' schools. In order to adapt the book especially for school use, questions have been added at the end of each chapter. "Clear, compact, and comprehensive."—DAILY NEWS. "*The relations of capital and labour have never been more simply or more clearly expounded.*—CONTEMPORARY REVIEW.

Elementary Class-Books—*continued.*

LOGIC.

ELEMENTARY LESSONS IN LOGIC; Deductive and Inductive, with copious Questions and Examples, and a Vocabulary of Logical Terms. By W. STANLEY JEVONS, M.A., Professor of Logic in Owens College, Manchester. Second Edition. 18mo. 3*s*. 6*d*.

In preparing these Lessons the author has attempted to show that Logic, even in its traditional form, can be made a highly useful subject of study, and a powerful means of mental exercise. With this view he has avoided the use of superfluous technical terms, and has abstained from entering into questions of a purely speculative or metaphysical character. For the puerile illustrations too often found in works on Logic, examples drawn from the distinct objects and ideas treated in the natural and experimental sciences have been generally substituted. At the end of almost every Lesson will be found references to the works in which the student will most profitably continue his reading of the subject treated, so that this little volume may serve as a guide to a more extended course of study. The GUARDIAN *thinks "nothing can be better for a school-book," and the* ATHENÆUM *calls it "a manual alike simple, interesting, and scientific."*

PHYSICS.

LESSONS IN ELEMENTARY PHYSICS. By BALFOUR STEWART, F.R.S., Professor of Natural Philosophy in Owens College, Manchester. With numerous Illustrations and Chromoliths of the Spectra of the Sun, Stars, and Nebulæ. Fifth Thousand. 18mo. 4*s*. 6*d*.

A description, in an elementary manner, of the most important of those laws which regulate the phenomena of nature. The active agents, heat, light, electricity, etc., are regarded as varieties of energy, and the work is so arranged that their relation to one another, looked at in this light, and the paramount importance of the laws of energy are clearly brought out. The volume contains all the necessary illustrations, and a plate representing the Spectra of Sun, Stars, and Nebulæ, forms a frontispiece. The EDUCATIONAL TIMES *calls this "the beau ideal of a scientific text-book, clear, accurate, and thorough."*

MANUALS FOR STUDENTS.

Flower (W. H.)—AN INTRODUCTION TO THE OSTEOLOGY OF THE MAMMALIA. Being the substance of the Course of Lectures delivered at the Royal College of Surgeons of England in 1870. By W. H. FLOWER, F.R.S., F.R.C.S., Hunterian Professor of Comparative Anatomy and Physiology. With numerous Illustrations. Globe 8vo. 7*s*. 6*d*.

Although the present work contains the substance of a Course of Lectures, the form has been changed, so as the better to adapt it as a handbook for students. Theoretical views have been almost entirely excluded: and while it is impossible in a scientific treatise to avoid the employment of technical terms, it has been the author's endeavour to use no more than absolutely necessary, and to exercise due care in selecting only those that seem most appropriate, or which have received the sanction of general adoption. With a very few exceptions the illustrations have been drawn expressly for this work from specimens in the Museum of the Royal College of Surgeons.

Hooker (Dr.)—THE STUDENT'S FLORA OF THE BRITISH ISLANDS. By J. D. HOOKER, C.B., F.R.S., M.D., D.C.L., Director of the Royal Gardens, Kew. Globe 8vo. 10*s*. 6*d*.

The object of this work is to supply students and field-botanists with a fuller account of the Plants of the British Islands than the manuals hitherto in use aim at giving. The Ordinal, Generic, and Specific characters have been re-written, and are to a great extent original, and drawn from living or dried specimens, or both. "*Cannot fail to perfectly fulfil the purpose for which it is intended.*"—LAND AND WATER. "*Containing the fullest and most accurate manual of the kind that has yet appeared.*"—PALL MALL GAZETTE.

Oliver (Professor).—FIRST BOOK OF INDIAN BOTANY. By DANIEL OLIVER, F.R.S., F.L.S., Keeper of the Herbarium and Library of the Royal Gardens, Kew, and Professor of Botany in University College, London. With numerous Illustrations. Extra fcap. 8vo. 6*s*. 6*d*.

This manual is, in substance, the author's "Lessons in Elementary Botany," adapted for use in India. In preparing it he has had in view the want, often felt, of some handy résumé of Indian Botany, which might be serviceable not only to residents of India, but also to any one about to proceed thither, desirous of getting some preliminary idea of the Botany of that country. "It contains a well-digested summary of all essential knowledge pertaining to Indian botany, wrought out in accordance with the best principles of scientific arrangement."—ALLEN'S INDIAN MAIL.

Other volumes of these Manuals will follow.

Ball (R. S., A.M.)—EXPERIMENTAL MECHANICS. A Course of Lectures delivered at the Royal College of Science for Ireland. By ROBERT STAWELL BALL, A.M., Professor of Applied Mathematics and Mechanics in the Royal College of Science for Ireland (Science and Art Department). Royal 8vo. 16*s*.

These twenty Lectures, delivered by the author in the spring of 1870, *have in the present volume been revised, and some of them rewritten. His aim has been to create in the mind of the student physical ideas corresponding to theoretical laws, and thus to produce a work which may be regarded either as a supplement or an introduction to manuals of theoretic mechanics. To realize this design, the copious use of experimental illustrations was necessary. The apparatus used in the Lectures, and figured in the volume, has been principally built up from Professor Willis's most admirable system. In the selection of the subjects, the question of practical utility has in many cases been regarded as the one of paramount* *importance. The elementary truths of Mechanics are too well known to admit of novelty, but it is believed that the mode of treatment which is adopted is more or less original. This is especially the case in the Lectures relating to friction, to the mechanical powers, to the strength of timber and structures, to the laws of motion, and to the pendulum. The illustrations, drawn from the apparatus, are nearly all original, and are beautifully executed.*

Cooke (Josiah P., Jun.).—FIRST PRINCIPLES OF CHEMICAL PHILOSOPHY. By JOSIAH P. COOKE, Jun., Ervine Professor of Chemistry and Mineralogy in Harvard College. Crown 8vo. 12*s.*

The object of the author in this book is to present the philosophy of Chemistry in such a form that it can be made with profit the subject of College recitations, and furnish the teacher with the means of testing the student's faithfulness and ability. With this view the subiect has been developed in a logical order, and the principles of the science are taught independently of the experimental evidence on which they rest.

Roscoe (H. E.)—SPECTRUM ANALYSIS. Six Lectures, with Appendices, Engravings, Maps, and Chromolithographs. By H. E. ROSCOE, F.R.S., Professor of Chemistry in Owens College, Manchester. Royal 8vo. 21*s.*

A Second Edition of these popular Lectures, containing all the most recent discoveries and several additional Illustrations. "In six lectures he has given the history of the discovery and set forth the facts relating to the analysis of light in such a way that any reader of ordinary intelligence and information will be able to understand what 'Spectrum Analysis' is, and what are its claims to rank among the most signal triumphs of science of which even this century can boast."—NONCONFORMIST. *"The illustrations—no unimportant part of a book on such a subject—are marvels of wood-printing, and reflect the clearness which is the distinguishing merit of Mr. Roscoe's explanations."*—SATURDAY REVIEW. *"The lectures themselves furnish a most ad-*

mirable elementary treatise on the subject, whilst by the insertion in appendices to each lecture of extracts from the most important published memoirs, the author has rendered it equally valuable as a text book for advanced students."—WESTMINSTER REVIEW.

Thorpe (T. E.)—A SERIES OF CHEMICAL PROBLEMS, for use in Colleges and Schools. Adapted for the preparation of Students for the Government, Science, and Society of Arts Examinations. With a Preface by Professor ROSCOE. 18mo. cloth. 1*s.*

In the Preface Dr. Roscoe says—" My experience has led me to feel more and more strongly that by no method can accuracy in a knowledge of chemistry be more surely secured than by attention to the working of well-selected problems, and Dr. Thorpe's thorough acquaintance with the wants of the student is a sufficient guarantee that this selection has been carefully made. I intend largely to use these questions in my own classes, and I can confidently recommend them to all teachers and students of the science."

Wurtz.—A HISTORY OF CHEMICAL THEORY, from the Age of Lavoisier down to the present time. By AD. WURTZ. Translated by HENRY WATTS, F.R.S. Crown 8vo. 6*s.*

"*The treatment of the subject is admirable, and the translator has evidently done his duty most efficiently.*"—WESTMINSTER REVIEW. "*The discourse, as a* résumé *of chemical theory and research, unites singular luminousness and grasp. A few judicious notes are added by the translator.*"—PALL MALL GAZETTE.

MISCELLANEOUS.

Abbott.—A SHAKESPEARIAN GRAMMAR. An Attempt to illustrate some of the Differences between Elizabethan and Modern English. By the Rev. E. A. ABBOTT, M.A., Head Master of the City of London School. For the Use of Schools. New and Enlarged Edition. Extra fcap. 8vo. 6*s.*

The object of this work is to furnish students of Shakespeare and Bacon with a short systematic account of some points of difference between Elizabethan syntax and our own. A section on Prosody is added, and Notes and Questions. The success which has attended the First and Second Editions of the "SHAKESPEARIAN GRAMMAR," *and the demand for a Third Edition within a year of the publication of the First, has encouraged the Author to endeavour to make the work somewhat more useful, and to render it, as far as possible, a complete book of reference for all difficulties of Shakespearian syntax or prosody. For this purpose the whole of Shakespeare has been re-read, and an attempt has been made to include within this Edition the explanation of every idiomatic difficulty* (*where the text is not confessedly corrupt*) *that comes within the province of a grammar as distinct from a glossary. The great object being to make a useful book of reference for students, and especially for classes in schools, several Plays have been indexed so fully that with the aid of a glossary and historical notes the references will serve for a complete commentary.* "*A critical inquiry, conducted with great skill and knowledge, and with all the appliances of modern philology . . . We venture to believe that those who consider themselves most proficient as Shakespearians will find something to learn from its pages.*"—PALL MALL GAZETTE. "*Valuable not only as an aid to the critical study of Shakespeare, but as tending to familiarize the reader with Elizabethan English in general.*"—ATHENÆUM.

Atlas of Europe. GLOBE EDITION. Uniform in size with Macmillan's Globe Series, containing 45 Coloured Maps, on a uniform scale and projection: with Plans of London and Paris, and a copious Index. Strongly bound in half-morocco, with flexible back, 9*s.*

This Atlas includes all the countries of Europe in a series of 48 *Maps, drawn on the same scale, with an Alphabetical Index to the situation of more than ten thousand places; and the relation of the various maps and countries to each other is defined in a general Key-map. The identity of scale in all the maps facilitates the comparison of extent and distance, and conveys a just impression of the magnitude of different countries. The size suffices to show the provincial divisions, the railways and main roads, the principal rivers and mountain ranges. "This Atlas," writes the* British Quarterly, "*will be an invaluable boon for the school, the desk, or the traveller's portmanteau.*"

Bates.—A CLASS-BOOK OF GEOGRAPHY. Adapted to the recent Programme of the Royal Geographical Society. By H. W. BATES, Assistant Secretary to the Royal Geographical Society. [*In the Press.*

CAMEOS FROM ENGLISH HISTORY.—See YONGE (C. M.).

Delamotte.—A BEGINNER'S DRAWING BOOK. By P. H. DELAMOTTE, F.S.A. Progressively arranged, with upwards of Fifty Plates. Crown 8vo. Stiff covers. 2*s.* 6*d.*

This work is intended to give such instruction to Beginners in Drawing, and to place before them copies so easy, that they may not find any obstacle in making the first step. Thenceforward the lessons are gradually progressive. Mechanical improvements too have lent their aid. The whole of the Plates have been engraved by a new process, by means of which a varying depth of tone—up to the present time the distinguishing characteristic of pencil drawing—has been imparted to woodcuts. "We have seen and examined a great many drawing-books, but the one now before us strikes us as being the best of them all."—ILLUSTRATED TIMES. "*A concise, simple, and thoroughly practical work. The letter-press is throughout intelligible and to the point.*"—GUARDIAN.

D'Oursy and Feillet.—A FRENCH GRAMMAR AT SIGHT, on an entirely new method. By A. D'OURSY and A. FEILLET. Especially adapted for Pupils preparing for Examination. Fcap. 8vo. cloth extra. 2*s.* 6*d.*

The method followed in this volume consists in presenting the grammar as much as possible by synoptical tables, which, striking the eye at once, and following throughout the same order—"used—not used;" "changes—does not change"*—are easily remembered. The parsing tables will enable the pupil to parse easily from the beginning. The exercises consist of translations from French into English, and from English into French; and of a number of grammatical questions.*

Freeman (Edward A.)—OLD-ENGLISH HISTORY. By EDWARD A. FREEMAN, D.C.L., late Fellow of Trinity College, Oxford. With Five Coloured Maps. Second Edition. Extra fcap. 8vo. half-bound. 6*s.*

The first edition of this work was an experiment, but an experiment which the Author found had already succeeded with his own children. The rapid sale of the first edition and the universal approval with which it has been received, show that the Author's convictions have been well founded, that his views have been widely accepted both by teachers and learners, and that the work is eminently calculated to serve the purpose for which it was intended. Although full of instruction and calculated highly to interest and even fascinate children, it is a work which may be and has been used with profit and pleasure by all. Its object is to show that clear, accurate, and scientific views of history, or indeed of any subject, may be easily given to children from the very first. "I have, I hope," the Author says, "shown that it is perfectly easy to teach children, from the very first, to distinguish true history alike from legend and from wilful invention, and also to understand the nature of historical authorities and to weigh one statement against another. I have throughout striven to connect the history of England with the general history of civilized Europe, and I have especially tried to make the book serve as an incentive to a more accurate study of historical geography." In the present edition the whole has been carefully revised, and such improvements as suggested *themselves have been introduced. "The book indeed is full of instruction and interest to students of all ages, and he must be a well-informed man indeed who will not rise from its perusal with clearer and more accurate ideas of a too much neglected portion of English History."*—SPECTATOR.

Helfenstein (James).—A COMPARATIVE GRAMMAR OF THE TEUTONIC LANGUAGES. Being at the same time a Historical Grammar of the English Language, and comprising Gothic, Anglo-Saxon, Early English, Modern English, Icelandic (Old Norse), Danish, Swedish, Old High German, Middle High German, Modern German, Old Saxon, Old Frisian, and Dutch. By JAMES HELFENSTEIN, Ph.D. 8vo. 18*s.*

This work traces the different stages of development through which the various Teutonic languages have passed, and the laws which have regulated their growth. The reader is thus enabled to study the relation which these languages bear to one another, and to the English language in particular, to which special attention is devoted throughout. In the chapters on Ancient and Middle Teutonic languages no grammatical form is omitted the knowledge of which is required for the study of ancient literature, whether Gothic or Anglo-Saxon or Early English. To each chapter is prefixed a sketch showing the relation of the Teutonic to the cognate languages, Greek, Latin, and Sanskrit. Those who have mastered the book will be in a position to proceed with intelligence to the more elaborate works of Grimm, Bopp, Pott, Schleicher, and others.

Hole.—A GENEALOGICAL STEMMA OF THE KINGS OF ENGLAND AND FRANCE. By the Rev. C. HOLE. On Sheet. 1*s.*

The different families are printed in distinguishing colours, thus facilitating reference.

A BRIEF BIOGRAPHICAL DICTIONARY. Compiled and Arranged by CHARLES HOLE, M.A., Trinity College, Cambridge. Second Edition, 18mo. neatly and strongly bound in cloth. 4*s.* 6*d.*

The inquiry is frequently made concerning an eminent man, when did he live, or for what was he celebrated, or what biographies have we about him? Such information is concisely supplied in this Dictionary. It contains more than 18,000 *names. Extreme care has been bestowed on the verifica-*

tion of the dates, and thus numerous errors current in previous works have been corrected. Its size adapts it for the desk, portmanteau, or pocket. "An invaluable addition to our manuals of reference, and from its moderate price cannot fail to become as popular as it is useful."—TIMES.

Jephson.—SHAKESPEARE'S "TEMPEST." With Glossarial and Explanatory Notes. By the Rev. J. M. JEPHSON. 18mo. 1*s.* 6*d.*

It is important to find some substitute for classical study, and it is believed that such a substitute may be found in the Plays of Shakespeare. Each sentence of Shakespeare becomes, like a sentence in Thucydides or Cicero, a lesson in the origin and derivation of words, and in the fundamental rules of grammatical construction. On this principle the present edition of the "Tempest" has been prepared. The text is taken from the "Cambridge Shakespeare."

M'Cosh (Rev. Principal).—For other Works by the same Author *see* PHILOSOPHICAL CATALOGUE.

THE LAWS OF DISCURSIVE THOUGHT. Being a Text-Book of Formal Logic. By JAMES M'COSH, D.D., LL.D. 8vo. 5*s.*

In this treatise the Notion (*with the Term and the Relation of Thought to Language,*) *will be found to occupy a larger relative place than in any logical work written since the time of the famous "Art of Thinking." "We heartily welcome his book as one which is likely to be of great value in Colleges and Schools."*—ATHENÆUM.

Oppen.—FRENCH READER. For the Use of Colleges and Schools. Containing a graduated Selection from modern Authors in Prose and Verse; and copious Notes, chiefly Etymological. By EDWARD A. OPPEN. Fcap. 8vo. cloth. 4*s.* 6*d.*

This is a Selection from the best modern authors of France. Its distinctive feature consists in its etymological notes, connecting French with the classical and modern languages, including the Celtic. This subject has hitherto been little discussed even by the best-educated teachers.

SHILLING BOOK OF GOLDEN DEEDS See YONGE (C. M.)

Sonnenschein and Meiklejohn.—THE ENGLISH METHOD OF TEACHING TO READ. By A. SONNENSCHEIN and J. M. D. MEIKLEJOHN, M.A. Fcap. 8vo.

COMPRISING

THE NURSERY BOOK, containing all the Two-Letter Words in the Language. 1*d.* (Also in Large Type on Sheets for School Walls. 5*s.*)

THE FIRST COURSE, consisting of Short Vowels with Single Consonants. 3*d.*

THE SECOND COURSE, with Combinations and Bridges, consisting of Short Vowels with Double Consonants. 4*d.*

THE THIRD AND FOURTH COURSES, consisting of Long Vowels, and all the Double Vowels in the Language. 6*d.*

A Series of Books in which an attempt is made to place the process of learning to read English on a scientific basis. This has been done by separating the perfectly regular parts of the language from the irregular, and by giving the regular parts to the learner in the exact order of their difficulty. The child begins with the smallest possible element, and adds to that element one letter—in only one of its functions—at one time. Thus the sequence is natural and complete. " These are admirable books, because they are constructed on a principle, and that the simplest principle on which it is possible to learn to read English."—SPECTATOR.

Vaughan (C. M.)—A SHILLING BOOK OF WORDS FROM THE POETS. By C. M. VAUGHAN. 18mo. cloth.

It has been felt of late years that the children of our parochial schools, and those classes of our countrymen which they commonly represent, are capable of being interested, and therefore benefited also, by something higher in the scale of poetical composition than those brief and somewhat puerile fragments to which their knowledge was formerly restricted. An attempt has here been made to supply the want by forming a selection at once various and unambitious; healthy in tone, just in sentiment, elevating in thought, and beautiful in expression.

Thring.—Works by EDWARD THRING, M.A., Head Master of Uppingham.

THE ELEMENTS OF GRAMMAR TAUGHT IN ENGLISH, with Questions. Fourth Edition. 18mo. 2*s.*

This little work is chiefly intended for teachers and learners. It took its rise from questionings in National Schools, and the whole of the first part is merely the writing out in order the answers to questions which have been used already with success. A chapter on Learning Language is especially addressed to teachers.

THE CHILD'S GRAMMAR. Being the Substance of "The Elements of Grammar taught in English," adapted for the Use of Junior Classes. A New Edition. 18mo. 1*s.*

SCHOOL SONGS. A Collection of Songs for Schools. With the Music arranged for four Voices. Edited by the Rev. E. THRING and H. RICCIUS. Folio. 7*s.* 6*d.*

There is a tendency in schools to stereotype the forms of life. Any genial solvent is valuable. Games do much; but games do not penetrate to domestic life, and are much limited by age. Music supplies the want. The collection includes the "Agnus Dei," Tennyson's "Light Brigade," Macaulay's "Ivry," &c. among other pieces.

Trench (Archbishop).—HOUSEHOLD BOOK OF ENGLISH POETRY. Selected and Arranged, with Notes, by R. C. TRENCH, D.D., Archbishop of Dublin. Extra fcap. 8vo. 5*s.* 6*d.* Second Edition.

This volume is called a "Household Book," by this name implying that it is a book for all—that there is nothing in it to prevent it from being confidently placed in the hands of every member of the household. Specimens of all classes of poetry are given, including selections from living authors. The Editor has aimed to produce a book "which the emigrant, finding room for little not absolutely necessary, might yet find room for it *in his trunk, and the traveller in his knapsack, and that on some narrow shelves where there are few books this might be one."* "*The Archbishop has conferred in this delightful volume an important gift on the whole English-speaking population of the world.*"—PALL MALL GAZETTE.

Yonge (Charlotte M.)—A PARALLEL HISTORY OF FRANCE AND ENGLAND: consisting of Outlines and Dates. By CHARLOTTE M. YONGE, Author of "The Heir of Redclyffe," "Cameos of English History," &c., &c. Oblong 4to. 3*s.* 6*d.*

This tabular history has been drawn up to supply a want felt by many teachers of some means of making their pupils realize what events in the two countries were contemporary. A skeleton narrative has been constructed of the chief transactions in either country, placing a column between for what affected both alike, by which means it is hoped that young people may be assisted in grasping the mutual relation of events. "*We can imagine few more really advantageous courses of historical study for a young mind than going carefully and steadily through Miss Yonge's excellent little book.*"—EDUCATIONAL TIMES.

CAMEOS FROM ENGLISH HISTORY. From Rollo to Edward II. By the Author of "The Heir of Redclyffe." Extra fcap. 8vo. Second Edition, enlarged. 5*s.*

The endeavour has not been to chronicle facts, but to put together a series of pictures of persons and events, so as to arrest the attention, and give some individuality and distinctness to the recollection, by gathering together details at the most memorable moments. The "Cameos" are intended as a book for young people just beyond the elementary histories of England, and able to enter in some degree into the real spirit of events, and to be struck with characters and scenes presented in some relief. "*Instead of dry details,*" *says the* Nonconformist, "*we have living pictures, faithful, vivid, and striking.*"

A SECOND SERIES OF CAMEOS FROM ENGLISH HISTORY. THE WARS IN FRANCE. Extra fcap. 8vo. pp. xi. 415. 5*s.*

This new volume, closing with the Treaty of Arras, is the history of the struggles of Plantagenet and Valois. It refers, accordingly, to one of the most stirring epochs in the mediæval era, including the battle of Poictiers,

Yonge (Charlotte M.)—*continued.*

the great Schism of the West, the Lollards, Agincourt and Joan of Arc. The Authoress reminds her readers that she aims merely at "collecting from the best authorities such details as may present scenes and personages to the eye in some fulness;" her CAMEOS *are a "collection of historical scenes and portraits such as the young might find it difficult to form for themselves without access to a very complete library." "Though mainly intended," says the* JOHN BULL, *"for young readers, they will, if we mistake not, be found very acceptable to those of more mature years, and the life and reality imparted to the dry bones of history cannot fail to be attractive to readers of every age."*

EUROPEAN HISTORY. Narrated in a Series of Historical Selections from the Best Authorities. Edited and arranged by E. M. SEWELL and C. M. YONGE. First Series, 1003—1154. Crown 8vo. 6*s.* Second Series, 1088—1228. Crown 8vo. 6*s.*

When young children have acquired the outlines of History from abridgments and catechisms, and it becomes desirable to give a more enlarged view of the subject, in order to render it really useful and interesting, a difficulty often arises as to the choice of books. Two courses are open, either to take a general and consequently dry history of facts, such as Russel's Modern Europe, or to choose some work treating of a particular period or subject, such as the works of Macaulay and Froude. The former course usually renders history uninteresting; the latter is unsatisfactory because it is not sufficiently comprehensive. To remedy this difficulty selections, continuous and chronological, have, in the present volume, been taken from the larger works of Freeman, Milman, Palgrave, and others, which may serve as distinct landmarks of historical reading. "We know of scarcely anything," says the GUARDIAN *of this volume, "which is so likely to raise to a higher level the average standard of English education."*

A SHILLING BOOK OF GOLDEN DEEDS. A Reading Book for Schools and General Readers. By the Author of "The Heir of Redclyffe." 18mo. cloth.

A record of some of the good and great deeds of all time, abridged from the larger work of the same author in the Golden Treasury Series.

DIVINITY.

⁂ For other Works by these Authors see THEOLOGICAL CATALOGUE.

Abbott (Rev. E. A.)—BIBLE LESSONS. By the Rev. E. A. ABBOTT, M.A., Head Master of the City of London School. Second Edition, crown 8vo. 4*s*. 6*d*.

This book is written in the form of dialogues carried on between a teacher and pupil, and its main object is to make the scholar think for himself. The great bulk of the dialogues represents in the spirit, and often in the words, the religious instruction which the author has been in the habit of giving to the Fifth and Sixth Forms of the City of London School. The Author has endeavoured to make the dialogues thoroughly unsectarian. "Wise, suggestive, and really profound initiation into religious thought."—GUARDIAN. *"I think nobody could read them without being both the better for them himself, and being also able to see how this difficult duty of imparting a sound religious education may be effected."—From* BISHOP OF ST. DAVID'S SPEECH AT THE EDUCATION CONFERENCE AT ABERGWILLY.

Cheyne (T. K.)—THE BOOK OF ISAIAH CHRONOLOGICALLY ARRANGED. An Amended Version, with Historical and Critical Introductions and Explanatory Notes. By T. K. CHEYNE, M.A., Fellow of Balliol College, Oxford. Crown 8vo. 7*s*. 6*d*.

The object of this edition is simply to restore the probable meaning of Isaiah, so far as this can be expressed in modern English. The basis of

---

the version is the revised translation of 1611, *but no scruple has been felt in introducing alterations, wherever the true sense of the prophecies appeared to require it.* "*A piece of scholarly work, very carefully and considerately done.*"—WESTMINSTER REVIEW.

Golden Treasury Psalter.—Students' Edition. Being an Edition of "The Psalms Chronologically Arranged, by Four Friends," with briefer Notes. 18mo. 3*s.* 6*d.*

In making this abridgment of "The Psalms Chronologically Arranged," the editors have endeavoured to meet the requirements of readers of a different class from those for whom the larger edition was intended. Some who found the large book useful for private reading, have asked for an edition of a smaller size and at a lower price, for family use, while at the same time some Teachers in Public Schools have suggested that it would be convenient for them to have a simpler book, which they could put into the hands of younger pupils. "*It is a gem,*" *says the* NONCONFORMIST.

Hardwick.—A HISTORY OF THE CHRISTIAN CHURCH. Middle Age. From Gregory the Great to the Excommunication of Luther. By ARCHDEACON HARDWICK. Edited by FRANCIS PROCTER, M.A. With Four Maps constructed for this work by A. KEITH JOHNSTON. Second Edition. Crown 8vo. 10*s.* 6*d.*

The ground-plan of this treatise coincides in many points with one adopted at the close of the last century in the colossal work of Schröckh, and since that time by others of his thoughtful countrymen; but in arranging the materials a very different course has frequently been pursued. With regard to the opinions of the author, he is willing to avow distinctly that he has construed history with the specific prepossessions of an Englishman and a member of the English Church. The reader is constantly referred to the authorities, both original and critical, on which the statements are founded. "*As a Manual for the student of ecclesiastical history in the Middle Ages, we know no English work which can be compared to Mr. Hardwick's book.*"—GUARDIAN.

Hardwick—*continued.*

A HISTORY OF THE CHRISTIAN CHURCH DURING THE REFORMATION. By ARCHDEACON HARDWICK. Revised by FRANCIS PROCTER, M.A. Second Edition. Crown 8vo. 10*s*. 6*d*.

This volume is intended as a sequel and companion to the "History of the Christian Church during the Middle Age." The author's earnest wish has been to give the reader a trustworthy version of those stirring incidents which mark the Reformation period, without relinquishing his former claim to characterise peculiar systems, persons, and events according to the shades and colours they assume, when contemplated from an English point of view, and by a member of the Church of England.

Maclear.—Works by the Rev. G. F. MACLEAR, B.D., Head Master of King's College School, and Preacher at the Temple Church.

A CLASS-BOOK OF OLD TESTAMENT HISTORY. Fifth Edition, with Four Maps. 18mo. cloth. 4*s*. 6*d*.

This volume forms a Class-book of Old Testament History from the earliest times to those of Ezra and Nehemiah. In its preparation the most recent authorities have been consulted, and wherever it has appeared useful, Notes have been subjoined illustrative of the Text, and, for the sake of more advanced students, references added to larger works. The Index has been so arranged as to form a concise dictionary of the persons and places mentioned in the course of the narrative; while the maps, which have been prepared with considerable care at Stanford's Geographical Establishment, will, it is hoped, materially add to the value and usefulness of the Book. "A careful and elaborate though brief compendium of all that modern research has done for the illustration of the Old Testament. We know of no work which contains so much important information in so small a compass."—BRITISH QUARTERLY REVIEW.

A CLASS-BOOK OF NEW TESTAMENT HISTORY, including the Connexion of the Old and New Testament. With Four Maps. Third Edition. 18mo. cloth. 5*s*. 6*d*.

A sequel to the author's Class-book of Old Testament History, continuing the narrative from the point at which it there ends, and carrying it on to

Maclear—*continued.*

the close of St. Paul's second imprisonment at Rome. In its preparation as in that of the former volume, the most recent and trustworthy authorities have been consulted, notes subjoined, and references to larger works added, It is thus hoped that it may prove at once an useful class-book and a convenient companion to the study of the Greek Testament. "A singularly clear and orderly arrangement of the Sacred Story. His work is solidly and completely done."—ATHENÆUM.

A SHILLING BOOK OF OLD TESTAMENT HISTORY, for National and Elementary Schools. With Map. 18mo. cloth.

A SHILLING BOOK OF NEW TESTAMENT HISTORY, for National and Elementary Schools. With Map. 18mo. cloth.

These works have been carefully abridged from the author's larger manuals.

CLASS-BOOK OF THE CATECHISM OF THE CHURCH OF ENGLAND. Second Edition. 18mo. cloth. 2*s.* 6*d.*

This may be regarded as a sequel to the Class-books of Old and New Testament History. Like them, it is furnished with notes and references to larger works, and it is hoped that it may be found, especially in the higher forms of our Public Schools, to supply a suitable manual of instruction in the chief doctrines of the English Church, and a useful help in the preparation of Candidates for Confirmation. "It is indeed the work of a scholar and divine, and as such, though extremely simple, it is also extremely instructive. There are few clergymen who would not find it useful in preparing candidates for Confirmation; and there are not a few who would find it useful to themselves as well."—LITERARY CHURCHMAN.

A FIRST CLASS-BOOK OF THE CATECHISM OF THE CHURCH OF ENGLAND, with Scripture Proofs, for Junior Classes and Schools. 18mo. 6*d.*

THE ORDER OF CONFIRMATION. A Sequel to the Class Book of the Catechism. For the use of Candidates for Confirmation. With Prayers and Collects. 18mo. 3*d.*

Maurice.—THE LORD'S PRAYER, THE CREED, AND THE COMMANDMENTS. A Manual for Parents and Schoolmasters. To which is added the Order of the Scriptures. By the Rev. F. DENISON MAURICE, M.A. Professor of Moral Philosophy in the University of Cambridge. 18mo. cloth limp. 1*s*.

Procter.—A HISTORY OF THE BOOK OF COMMON PRAYER, with a Rationale of its Offices. By FRANCIS PROCTER, M.A. Ninth Edition, revised and enlarged. Crown 8vo. 10*s*. 6*d*.

In the course of the last twenty years the whole question of Liturgical knowledge has been reopened with great learning and accurate research; and it is mainly with the view of epitomizing extensive publications, and correcting the errors and misconceptions which had obtained currency, that the present volume has been put together. "We admire the Author's diligence, and bear willing testimony to the extent and accuracy of his reading. The origin of every part of the Prayer Book has been diligently investigated, and there are few questions of facts connected with it which are not either sufficiently explained, or so referred to that persons interested may work out the truth for themselves."—ATHENÆUM.

Procter and Maclear.—AN ELEMENTARY INTRODUCTION TO THE BOOK OF COMMON PRAYER. Re-arranged and supplemented by an Explanation of the Morning and Evening Prayer and the Litany. By the Rev. F. PROCTER and the Rev. G. F. MACLEAR. Fourth Edition. 18mo. 2*s*. 6*d*.

As in the other Class-books of the series, notes have also been subjoined, and references given to larger works, and it is hoped that the volume will be found adapted for use in the higher forms of our Public Schools, and a suitable manual for those preparing for the Oxford and Cambridge local examinations. This new Edition has been considerably altered, and several important additions have been made. Besides a re-arrangement of the work generally, the Historical Portion has been supplemented by an Explanation of the Morning and Evening Prayer and of the Litany.

Psalms of David chronologically arranged. By Four Friends. An Amended Version, with Historical Introduction and Explanatory Notes. Second Edition, with Additions and Corrections. Crown 8vo. 8*s.* 6*d.*

To restore the Psalter as far as possible to the order in which the Psalms were written,—to give the division of each Psalm into strophes, of each strophe into the lines which composed it,—to amend the errors of translation, is the object of the present Edition. Professor Ewald's works, especially that on the Psalms, have been extensively consulted. This book has been used with satisfaction by masters for private work in higher classes in schools. THE SPECTATOR *calls this "one of the most instructive and valuable books that has been published for many years."*

Ramsay.—THE CATECHISER'S MANUAL; or, the Church Catechism illustrated and explained, for the use of Clergymen, Schoolmasters, and Teachers. By the Rev. ARTHUR RAMSAY, M.A. Second Edition. 18mo. 1*s.* 6*d.*

A clear explanation of the Catechism, by way of Question and Answer. "This is by far the best Manual on the Catechism we have met with." —ENGLISH JOURNAL OF EDUCATION.

Simpson.—AN EPITOME OF THE HISTORY OF THE CHRISTIAN CHURCH. By WILLIAM SIMPSON, M.A. Fifth Edition. Fcap. 8vo. 3*s.* 6*d.*

A compendious summary of Church History.

Swainson.—A HANDBOOK to BUTLER'S ANALOGY. By C. A. SWAINSON, D.D., Canon of Chichester. Crown 8vo. 1*s.* 6*d.*

This manual is designed to serve as a handbook or road-book to the Student in reading the Analogy, to give the Student a sketch or outline map of the country on which he is entering, and to point out to him matters of interest as he passes along.

Westcott.—Works by BROOKE FOSS WESTCOTT, B.D., Canon of Peterborough.

A GENERAL SURVEY OF THE HISTORY OF THE CANON OF THE NEW TESTAMENT DURING THE FIRST FOUR CENTURIES. Third Edition, revised. Crown 8vo. 10*s.* 6*d.*

The author has endeavoured to connect the history of the New Testament Canon with the growth and consolidation of the Church, and to point out the relation existing between the amount of evidence for the authenticity of its component parts, and the whole mass of Christian literature. Such a method of inquiry will convey both the truest notion of the connexion of the written Word with the living Body of Christ, and the surest conviction of its divine authority. Of this work the SATURDAY REVIEW *writes: "Theological students, and not they only, but the general public, owe a deep debt of gratitude to Mr. Westcott for bringing this subject fairly before them in this candid and comprehensive essay. . . . As a theological work it is at once perfectly fair and impartial, and imbued with a thoroughly religious spirit; and as a manual it exhibits, in a lucid form and in a narrow compass, the results of extensive research and accurate thought. We cordially recommend it."*

NTRODUCTION TO THE STUDY OF THE FOUR GOSPELS. Third Edition. Crown 8vo. 10*s.* 6*d.*

This book is intended to be an Introduction to the Study of the Gospels. The author has made it a point carefully to study the researches of the great writers, and consciously to neglect none. There is an elaborate discussion appended "On the Primitive Doctrine of Inspiration." "His 'Introduction' and 'Canon' are two of the best works of the kind to be found in any literature."—DAILY NEWS.

A GENERAL VIEW OF THE HISTORY OF THE ENGLISH BIBLE. Crown 8vo. 10*s.* 6*d.*

"The first trustworthy account we have had of that unique and marvellous monument of the piety of our ancestors."—DAILY NEWS.

Westcott (Canon)—*continued.*

"*A brief, scholarly, and, to a great extent, an original contribution to theological literature. He is the first to offer any considerable contributions to what he calls their internal history, which deals with their relation to other texts, with their filiation one on another, and with the principles by which they have been successively modified.*"—PALL MALL GAZETTE.

THE BIBLE IN THE CHURCH. A Popular Account of the Collection and Reception of the Holy Scriptures in the Christian Churches. Third Edition. 18mo. cloth, 4*s.* 6*d.*

The present book is an attempt to answer a request, which has been made from time to time, to place in a simple form, for the use of general readers, the substance of the author's "History of the Canon of the New Testament." An elaborate and comprehensive Introduction is followed by chapters on the Bible of the Apostolic Age; on the Growth of the New Testament; the Apostolic Fathers; the Age of the Apologists; the First Christian Bible; the Bible Proscribed and Restored; the Age of Jerome and Augustine; the Bible of the Middle Ages in the West and in the East, and in the Sixteenth Century. Two appendices on the History of the Old Testament Canon before the Christian Era, and on the Contents of the most ancient MSS. of the Christian Bible, complete the volume. "We would recommend every one who loves and studies the Bible to read and ponder this exquisite little book. Mr. Westcott's account of the 'Canon' is true history *in its highest sense.*"—LITERARY CHURCHMAN.

THE GOSPEL OF THE RESURRECTION. Thoughts on its Relation to Reason and History. New Edition. Fcap. 8vo. 4*s.* 6*d.*

This Essay is an endeavour to consider some of the elementary truths of Christianity as a miraculous Revelation, from the side of History and Reason. If the arguments which are here adduced are valid, they will go far to prove that the Resurrection, with all that it includes, is the key to the history of man, and the complement of reason.

Wilson.—THE BIBLE STUDENTS' GUIDE to the more Correct Understanding of the English translation of the Old Testament, by reference to the Original Hebrew. By WILLIAM WILSON, D.D., Canon of Winchester, late Fellow of Queen's College, Oxford. Second Edition, carefully Revised. 4to. cloth. 25*s.*

This work is the result of almost incredible labour bestowed on it during many years. Its object is to enable the readers of the Old Testament Scriptures to penetrate into the real meaning of the sacred writers. All the English words used in the Authorized Version are alphabetically arranged, and beneath them are given the Hebrew equivalents, with a careful explanation of the peculiar signification and construction of each term. The knowledge of the Hebrew language is not absolutely necessary to the profitable use of the work. Devout and accurate students of the Bible, entirely unacquainted with Hebrew, may derive great advantage from frequent reference to it. It is especially adapted for the use of the clergy. "For all earnest students of the Old-Testament Scriptures it is a most valuable Manual. Its arrangement is so simple that those who possess only their mother-tongue, if they will take a little pains, may employ it with great profit."—NONCONFORMIST.

Yonge (Charlotte M.)—SCRIPTURE READINGS FOR SCHOOLS AND FAMILIES. By CHARLOTTE M. YONGE, Author of "The Heir of Redclyffe." Globe 8vo. 1*s.* 6*d.* With Comments, 3*s.* 6*d.*

Every one engaged in education must at times have felt some difficulty on the subject of reading Holy Scripture with children. Actual need has led the author to endeavour to prepare a reading book convenient for study with children, containing the very words of the Bible, with only a few expedient omissions, and arranged in Lessons of such length as by experience she has found to suit with children's ordinary power of accurate attentive interest. The verse form has been retained, because of its convenience for children reading in class, and as more resembling their Bibles; but the poetical portions have been given in their lines. When Psalms or portions from the Prophets illustrate or fall in with the narrative they are *given in their chronological sequence. The Scripture portion, with a very few notes explanatory of mere words, is bound up apart, to be used by children, while the same is also supplied with a brief comment, the purpose of which is either to assist the teacher in explaining the lesson, or to be used by more advanced young people to whom it may not be possible to give access to the authorities whence it has been taken.*

Professor Huxley, at a meeting of the London School Board, particularly mentioned the selection made by Miss Yonge as an example of how selections might be made from the Bible for School reading. See TIMES, *March* 30, 1871.

BOOKS ON EDUCATION.

Arnold.—A FRENCH ETON; OR, MIDDLE CLASS EDUCATION AND THE STATE. By MATTHEW ARNOLD. Fcap. 8vo. cloth. 2*s*. 6*d*.

This interesting little volume is the result of a visit to France in 1859 *by Mr. Arnold, authorised by the Royal Commissioners, who were then inquiring into the state of popular education in England, to seek, in their name, information respecting the French Primary Schools. "A very interesting dissertation on the system of secondary instruction in France, and on the advisability of copying the system in England."*—SATURDAY REVIEW.

Jex-Blake.—A VISIT TO SOME AMERICAN SCHOOLS AND COLLEGES. By SOPHIA JEX-BLAKE. Crown 8vo. cloth. 6*s*.

"In the following pages I have endeavoured to give a simple and accurate account of what I saw during a series of visits to some of the Schools and Colleges in the United States. . . . I wish simply to give other teachers an opportunity of seeing through my eyes what they cannot perhaps see for themselves, and to this end I have recorded just such particulars as I should myself care to know."—AUTHOR'S PREFACE. *"Miss Blake gives a living picture of the Schools and Colleges themselves in which that education is carried on."*—PALL MALL GAZETTE.

Quain (Richard, F.R.S.)—ON SOME DEFECTS IN GENERAL EDUCATION. By RICHARD QUAIN, F.R.S. Crown 8vo. 3*s.* 6*d.*

Having been charged by the College of Surgeons with the delivery of the Hunterian Oration for 1869, *the Author has availed himself of the occasion to bring under notice some defects in the general education of the country, which, in his opinion, effect injuriously all classes of the people, and not least the members of his own profession. The earlier pages of the address contain a short notice of the genius and labours of John Hunter, but the subject of education will be found to occupy the larger part—from page twelve to the end.* THE EXAMINER *calls the work "an interesting addition to educational literature."*

Thring.—EDUCATION AND SCHOOL. By the Rev. EDWARD THRING, M.A., Head Master of Uppingham. Second Edition. Crown 8vo. cloth. 5*s.* 6*d.*

"*An invaluable book on a subject of the highest importance.*"—ENGLISH INDEPENDENT.

Youmans.—MODERN CULTURE: its True Aims and Requirements. A Series of Addresses and Arguments on the Claims of Scientific Education. Edited by EDWARD L. YOUMANS, M.D. Crown 8vo. 8*s.* 6*d.*

CONTENTS:—*Professor Tyndall on the Study of Physics; Dr. Daubeny on the Study of Chemistry; Professor Henfrey on the Study of Botany; Professor Huxley on the Study of Zoology; Dr. J. Paget on the Study of Physiology; Dr. Whewell on the Educational History of Science; Dr. Faraday on the Education of the Judgment; Dr. Hodgson on the Study of Economic Science; Mr. Herbert Spencer on Political Education; Professor Masson on College Education and Self Education; Dr. Youmans on the Scientific Study of Human Nature. An Appendix contains extracts from distinguished authors, and from the Scientific Evidence given before the Public Schools Commission.*

London: R. Clay, Sons, and Taylor, Printers.